c Differential calculus and Trigonometry
1st. stage (2nd. Sec.)

Answer all the following questions

1) a) Evaluate:-

(i) $\lim\limits_{x\to 8} \dfrac{\sqrt{7+\sqrt[3]{x}} - 3}{x - 8}$

(ii) $\lim\limits_{x\to\infty}\left(3-\dfrac{2}{x}\right)\left(\dfrac{3}{x}\sqrt{x^2+1}\right)$

(iii) $\lim\limits_{x\to 0}\dfrac{\sin 3x\,\square\,\tan^2 x}{x^2 \sin 7x}$

b) Find the first derivative of each the following

i) $f(x)=\left(\dfrac{x^2 + 2}{x^2 + 3}\right)^4$

ii) $y=x^3\sqrt[5]{(x^2+2x+2)^2}$

2) a) If $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}+ax+b\right)=0$

find the value a and b

b) Find the points on the curve of $y=\dfrac{x^2+x+1}{x+1}$ at which the tangents are perpendicular to the st. line $4x=5-3y$

3)a) If $y=\dfrac{z^2}{z^2+1}$, $z=\sqrt{2x+1}$ find $\dfrac{dy}{dx}$

b) prove that the two curves $y=x^2-x+2$ and $y = 3x - x^2$ are touch each other at their point of intersection then find the equation of their common tangent

4)

a) Without using calculator find the value of $\sin 15^\circ$, $\cos 75^\circ$, $\tan 105^\circ$

b) Evaluate

i) $\cos(100^\circ-x)\cos(70^\circ-x) + \sin(80^\circ+x)\sin(70^\circ-x)$

ii) $\sin 23^\circ \cos 7^\circ + \sin 67^\circ \cos 83^\circ$

5) a) If A , B are the measure of two acute angles such that
$m(\angle A)+m(\angle B)=120^0$ &
$2\sin A=(\sqrt{3}+1)\sin B$
Find A and B

b) If $0<x<2\pi$
and $\sin(60^\circ + x)=2\sin x$
Find the values of x

6)

a) If $m(\angle A)$, $m(\angle B)$, $m(\angle C)$, are the measures of three acute angles such that:

$\tan A=\dfrac{2}{3}$, $\tan B=\dfrac{1}{2}$, $\tan C=\dfrac{4}{7}$

without using calculator

prove that $A+B+C=\dfrac{\pi}{2}$

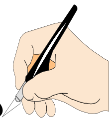
b)

i) Prove that:

$\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ$

ii) If tan A , tan B are the roots of the equation $4x^2-17x+15=0$
Calculate $\tan(A-B)$

The answers



1) a)

i) $f(x) = \dfrac{\sqrt{7+\sqrt[3]{x}}-3}{x-8}$

$f(8) = \dfrac{0}{0}$ unspecified value

$$\lim_{x\to 8}\frac{\sqrt{7+\sqrt[3]{x}}-3}{x-8} = \lim_{x\to 8}\frac{\sqrt{7+\sqrt[3]{x}}-3}{x-8}\times\frac{\sqrt{7+\sqrt[3]{x}}+3}{\sqrt{7+\sqrt[3]{x}}+3}$$

$$= \lim_{x\to 8}\frac{7+\sqrt[3]{x}-9}{(x-8)(\sqrt{7+\sqrt[3]{x}}+3)}$$

$$= \lim_{x\to 8}\frac{\sqrt[3]{x}-2}{(x-8)}\times\frac{1}{(\sqrt{7+\sqrt[3]{x}}+3)}$$

$$= \lim_{x\to 8}\frac{x^{\frac{1}{3}}-(8)^{\frac{1}{3}}}{x-8}\times\frac{1}{\sqrt{7\times\sqrt[3]{8}}+3}$$

$$= \frac{1}{3}(8)^{\frac{1}{3}-1}\times\frac{1}{\sqrt{7+2}+3}$$

$$= \frac{1}{3}(2^3)^{\frac{-2}{3}}\times\frac{1}{3+3} = \frac{1}{3}\times(2)^{-2}\times\frac{1}{6}$$

$$= \frac{1}{3}\times\frac{1}{4}\times\frac{1}{6} = \frac{1}{72}$$

ii) $f(x)=(3-\dfrac{2}{x})(\dfrac{3}{x}\sqrt{x^2+1})$

$f(\infty) = 0\times\infty$ unspecified value

$$\lim_{x\to\infty}\left(3-\frac{2}{x}\right)\left(\frac{3}{x}\sqrt{x^2+1}\right)$$

$$= \lim_{x\to\infty}(3-\frac{2}{x})\times\lim_{x\to\infty}\frac{3}{x}\sqrt{x^2+1}$$

$$= (3-\frac{2}{\infty})\times\lim_{x\to\infty}3\sqrt{\frac{1}{x^2}(x^2+1)}$$

$$= (3-0)\times\lim_{x\to\infty}3\sqrt{1+\frac{1}{x^2}}$$

$$= 3\times 3\sqrt{1+\frac{1}{\infty}}$$

$$= 3\times 3 = 9$$

iii) $f(x) = \dfrac{\sin 3x\tan^2 x}{x^2\sin 7x}$

$f(0) = \dfrac{0}{0}$ unspecified value

Divide up and down by x^3

$$\therefore \lim_{x\to 0}\frac{\sin 3x\tan^2 x}{x^2\sin 7x} = \lim_{x\to 0}\frac{\frac{\sin 3x}{x}\times\frac{\tan^2 x}{x^2}}{\frac{x^2\sin 7x}{x^3}}$$

$$= \frac{\lim_{x\to 0}\frac{\sin 3x}{x}\times\lim_{x\to 0}(\frac{\tan x}{x})^2}{\lim_{x\to 0}\frac{\sin 7x}{x}} = \frac{3x(1)^2}{7} = \frac{3}{7}$$

b) i) $f(x)=\left(\dfrac{x^2+2}{x^2+3}\right)^4$

$$f'(x)=4\left(\frac{x^2+2}{x^2+3}\right)^3\times\frac{(x^2+3)\times 2x-(x^2+2)\times 2x}{(x^2+3)^2}$$

$$=4\left(\frac{x^2+2}{x^2+3}\right)^3\times\frac{2x^3+6x-2x^3-4x}{(x^2+3)^2}$$

$$=4\times\frac{(x^2+2)^3}{(x^2+3)^3}\times\frac{2x}{(x^2+3)^2} = \frac{8x(x^2+2)^3}{(x^2+3)^5}$$

ii) $y=x^3\sqrt[5]{(x^2+2x+2)^2} = x^3(x^2+2x+2)^{\frac{2}{5}}$

$$\frac{dy}{dx}=x^3\times\frac{2}{5}(x^2+2x+2)^{\frac{-3}{5}}(2x+2)+$$

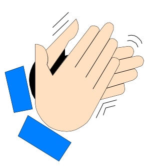
$$(x^2+2x+2)^{\frac{2}{5}}\times 3x^2$$

$$=\frac{2x^3(2x+2)}{5\sqrt[5]{(x^2+2x+2)^3}}+3x^2\sqrt[5]{(x^2+2x+2)^2}$$

$$=\frac{4x^3(x+1)}{5\sqrt[5]{(x^2+2x+2)^3}}+3x^2\sqrt[5]{(x^2+2x+2)^2}$$

$$=\frac{4x^3(x+1)+15x^2(x^2+2x+2)}{5\sqrt[5]{(x^2+2x+2)^3}}$$

$$=\frac{19x^4+34x^3+30x^2}{5\sqrt[5]{(x^2+2x+2)^3}}$$



2) a) $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x+1}+ax+b\right)=0$

$\therefore \lim\limits_{x\to\infty}\frac{x^2+1+(x+1)(ax+b)}{x+1}=0$

$\therefore \lim\limits_{x\to\infty}\frac{x^2+1+ax^2+bx+ax+b}{x+1}=0$

$\lim\limits_{x\to\infty}\frac{(1+a)x^2+(a+b)x+1+b}{x+1}=0$

The limit=zero at $x\to\infty$

Then the degree of numerator is smaller than the degree of denominator

$\therefore 1+a=0 \Rightarrow \boxed{a=-1}$

and $a+b=0 \Rightarrow b=-a \Rightarrow \boxed{b=1}$

b) $y=\frac{x^2+x+1}{x+1}$

$\frac{dy}{dx}=\frac{(x+1)(2x+1)-(x^2+x+1)\times 1}{(x+1)^2}$

$=\frac{2x^2+3x+1-x^2-x-1}{(x+1)^2}=\frac{x^2+2x}{(x+1)^2}$

$\because$ the tangent $\perp$ to st line $4x+3y-5=0$

$\therefore$ slope of tangent $=\frac{-1}{\text{slope of line}}$

$\therefore \frac{dy}{dx}=\frac{-1}{\frac{-\text{cofficient of } x}{\text{cofficient of } y}} \Rightarrow \frac{x^2+2x}{(x+1)^2}=\frac{-1}{\frac{-4}{3}}$

$\therefore \frac{x^2+2x}{x^2+2x+1}=\frac{3}{4} \Rightarrow 4x^2+8x=3x^2+6x+3$

$\therefore x^2+2x-3=0 \Rightarrow (x+3)(x-1)=0$

Either $x=-3$, $y=\frac{9-3+1}{-3+1}=\frac{-7}{2}$

or $x=1$, $y=\frac{1+1+1}{1+1}=\frac{3}{2}$

The points are $(-3,-\frac{7}{2})$, $(1,\frac{3}{2})$

3)

a) $y=\frac{z^2}{z^2+1}$

$\frac{dy}{dz}=\frac{(z^2+1)2z-z^2\times 2z}{(z^2+1)^2}$

$=\frac{\cancel{2z^3}+2z-\cancel{2z^3}}{(z^2+1)^2}$

$=\frac{2z}{(z^2+1)^2}$

$=\frac{2\sqrt{2x+1}}{(2x+1+1)^2}$

$=\frac{2\sqrt{2x+1}}{(2x+2)^2}$

$z=\sqrt{2x+1}$

$z=(2x+1)^{\frac{1}{2}}$

$\frac{dz}{dx}=\frac{1}{2}(2x+1)^{-\frac{1}{2}}\times 2$

$=\frac{1}{(2x+1)^{\frac{1}{2}}}$

$=\frac{1}{\sqrt{2x+1}}$

$\because \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx} \Rightarrow \frac{dy}{dx}=\frac{2\cancel{\sqrt{2x+1}}}{(2x+2)^2}\times\frac{1}{\cancel{\sqrt{2x+1}}}$

$\therefore \frac{dy}{dx}=\frac{2}{(2(x+1))^2}=\frac{2}{4(x+1)^2}=\frac{1}{2(x+1)^2}$

b) To find point of intersection of the two curves solve their equations

$y=x^2-x+2$ and $y=3x-x^2$

$\therefore x^2-x+2=3x-x^2 \Rightarrow 2x^2-4x+2=0 \ (\div 2)$

$\therefore x^2-2x+1=0 \Rightarrow (x-1)^2=0$

$x-1=0 \Rightarrow \boxed{x=1}$ then $y=1-1+2=2$

Then the point of intersection is (1,2)

$y=x^2-x+2$

$\frac{dy}{dx}=2x-1$

At $(1,2) \Rightarrow \frac{dy}{dx}=1$

& $y=3x-x^2$

$\frac{dy}{dx}=3-2x$

At $(1,2) \Rightarrow \frac{dy}{dx}=1$

$\because$ The two slopes are equals

$\therefore$ The Two tangents are parallel

$\because$ They have a common point (1,2)

$\therefore$ The Two tangents are Coinside

$\therefore$ The Two curves have a common tangent at (1,2)

$\therefore$ The equation of the tangent $\frac{y-y_1}{x-x_1}=\frac{dy}{dx}$

$\frac{y-2}{x-1}=1 \Rightarrow x-1=y-2 \Rightarrow x-y+1=0$

4. a)

$\sin 15 = \sin(45^\circ - 30^\circ)$

$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

$\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$

Notice that:

$\cos 75 = \cos(90^\circ - 15^\circ) = \sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$

$\tan 105^\circ = \tan(45^\circ + 60^\circ)$

$= \frac{\tan 45^\circ + \tan 60^\circ}{1 - \tan 45^\circ \tan 60^\circ} = \frac{1+\sqrt{3}}{1 - 1\times\sqrt{3}}$

$= \frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} = \frac{1+2\sqrt{3}+3}{1-3}$

$= \frac{4+2\sqrt{3}}{-2} = \frac{\cancel{2}(2+\sqrt{3})}{-\cancel{2}} = -(2+\sqrt{3})$

b) i) $\cos(100^\circ - x)\cos(70^\circ - x) + \sin(80^\circ + x)\sin(70^\circ - x)$

$= \cos(100^\circ - x)\cos(70^\circ - x) + \sin(180^\circ - (80^\circ + x))\sin(70^\circ - x)$

$= \cos(100^\circ - x)\cos(70^\circ - x) + \sin(100^\circ - x)\sin(70^\circ - x)$

$= \cos\left((100^\circ - x) - (70^\circ - x)\right)$

$= \cos\left(100^\circ - \cancel{x} - 70^\circ + \cancel{x}\right) = \cos(30^\circ) = \sqrt{3}/2$

ii) $\sin 23^\circ \cos 7^\circ + \sin 67^\circ \cos 83^\circ$

$\because \sin 67^\circ = \sin(90^\circ - 23^\circ) = \cos 23^\circ$

$\cos 83^\circ = \cos(90^\circ - 7^\circ) = \sin 7^\circ$

$\therefore \sin 23^\circ \cos 7^\circ + \sin 67^\circ \cos 83^\circ$

$= \sin 23^\circ \cos 7^\circ + \cos 23^\circ \sin 7^\circ$

$= \sin(23^\circ + 7^\circ) = \sin 30^\circ = \frac{1}{2}$

5) a) $\because A + B = 120^\circ \Rightarrow A = 120^\circ - B$

$\sin A = \sin(120^\circ - B)$

$\sin A = \sin 120^\circ \cos B - \cos 120^\circ \sin B$

$\therefore \sin A = \sin(180 - 60)\cos B - \cos(180 - 60)\sin B$

$= \sin 60 \cos B - (-\cos 60)\sin B$

$\therefore \sin A = \frac{\sqrt{3}}{2}\cos B + \frac{1}{2}\sin B$

Multiply both sides by 2

$\therefore 2\sin A = \sqrt{3}\cos B + \sin B \quad . . . . (1)$

From the given:

$\because 2\sin A = (\sqrt{3}+1)\sin B$ in (1)

$\therefore (\sqrt{3}+1)\sin B = \sqrt{3}\cos B + \sin B$

$\sqrt{3}\sin B + \cancel{\sin B} = \sqrt{3}\cos B + \cancel{\sin B}$

$\cancel{\sqrt{3}}\sin B = \cancel{\sqrt{3}}\cos B \Rightarrow \sin B = \cos B$

$\frac{\sin B}{\cos B} = \frac{\cos B}{\cos B} \Rightarrow \tan B = 1$

$, \angle B$ is an acute anqle

$\therefore m(B) = 45^0 \Rightarrow m(A) = 120^\circ - 45^\circ = 75^0$

b) $\because \sin(60^\circ + x) = 2\sin x$

$\therefore \sin 60 \cos x + \cos 60 \sin x = 2\sin x$

$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = 2\sin x$

$\frac{\sqrt{3}}{2}\cos x = \frac{3}{2}\sin x \Rightarrow \cos x = \sqrt{3}\sin x$

Divide both sides by cos x :-

$1 = \sqrt{3}\tan x \Rightarrow \tan x = \frac{1}{\sqrt{3}}$

Either $\boxed{x = 30^\circ}$ or $\boxed{x = 180^\circ + 30^\circ = 210^\circ}$

$\therefore S.S = \{30^\circ, 210^\circ\}$

6) a)

$$\because \tan(A+B)=\frac{\tan A+\tan B}{1-(\tan A\times\tan B)}$$

$$=\frac{\frac{2}{3}+\frac{1}{2}}{1-\left(\frac{\cancel{2}}{3}\times\frac{1}{\cancel{2}}\right)}=\frac{\frac{4+3}{6}}{1-\frac{1}{3}}=\frac{7}{6}\times\frac{3}{2}=\frac{7}{4}$$

$$\because \tan((A+B)+C)=\frac{\tan(A+B)+\tan C}{1-(\tan(A+B)\tan C)}$$

$$\frac{\frac{4}{4}+\frac{4}{7}}{1-\left(\frac{7}{4}\times\frac{4}{7}\right)}=\frac{\frac{49+16}{28}}{1-1}=\frac{\frac{65}{28}}{0}=\text{undfined}$$

$$\boxed{\therefore m(\angle A)+m(\angle B)+m(\angle C)=\frac{\pi}{2}}$$

Another method

$$\because \tan(A+B)=\frac{\tan A+\tan B}{1-(\tan A\times\tan B)}$$

$$=\frac{\frac{2}{3}+\frac{1}{2}}{1-\left(\frac{\cancel{2}}{3}\times\frac{1}{\cancel{2}}\right)}=\frac{\frac{4+3}{6}}{1-\frac{1}{3}}$$

$$=\frac{7}{6}\times\frac{3}{2}=\frac{7}{4}=\frac{1}{\tan C}=\cot C$$

$$\therefore \tan(A+B)=\cot C=\tan\left(\frac{\pi}{2}-C\right)$$

$$\therefore A+B=\frac{\pi}{2}-C$$

$$\therefore \boxed{m(\angle A)+m(\angle B)+m(\angle C)=\frac{\pi}{2}}$$

b)

i)

L.H.S

$=\tan 20^\circ\ \tan 40^\circ\ \tan 80^\circ$

$=\tan 20^\circ\ \tan(60^\circ-20^\circ)\ \tan(60^\circ+20^\circ)$

$$=\tan20^\circ\times\frac{\tan60^\circ-\tan20^\circ}{1+\tan60^\circ\tan20^\circ}\times\frac{\tan60^\circ+\tan20^\circ}{1-\tan60^\circ\tan20^\circ}$$

$$=\tan20^\circ\left(\frac{\sqrt{3}-\tan20^\circ}{1+\sqrt{3}\tan20^\circ}\times\frac{\sqrt{3}+\tan20^\circ}{1-\sqrt{3}\tan20^\circ}\right)$$

$$\boxed{\therefore \tan20^\circ\tan40^\circ\tan80=\tan20^\circ\times\frac{3-\tan^2 20^\circ}{1-3\tan^2 20^\circ}}$$

R.H.S $\because \tan60^\circ=\tan(20^\circ+40^\circ)$

$$\boxed{\therefore \tan60^\circ=\frac{\tan20^\circ+\tan40^\circ}{1-\tan20^\circ\tan40^\circ}}$$

$$\because \tan40^\circ=\frac{2\tan20^\circ}{1-\tan^2 20^\circ}$$

$$\therefore \tan20^\circ+\tan40^\circ=\tan20^\circ+\frac{2\tan20^\circ}{1-\tan^2 20^\circ}$$

$$=\tan20^\circ\left(1+\frac{2}{1-\tan^2 20^\circ}\right)$$

$$=\tan20^\circ\left(\frac{1-\tan^2 20^\circ}{1-\tan^2 20^\circ}+\frac{2}{1-\tan^2 20^\circ}\right)$$

$$\boxed{\therefore \tan20^\circ+\tan40^\circ=\tan20^\circ\left(\frac{3-\tan^2 20^\circ}{1-\tan^2 20^\circ}\right)\ldots(1)}$$

$$=1-\tan20^\circ\tan40^\circ=1-\left(\tan20^\circ\times\frac{2\tan20^\circ}{1-\tan^2 20^\circ}\right)$$

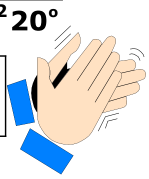
$$=1-\frac{2\tan^2 20^\circ}{1-\tan^2 20^\circ}=\frac{1-\tan^2 20^\circ}{1-\tan^2 20^\circ}-\frac{2\tan^2 20^\circ}{1-\tan^2 20^\circ}$$

$$\boxed{\therefore 1-\tan20^\circ\tan40^\circ=\frac{1-3\tan^2 20^\circ}{1-\tan^2 20^\circ}\ldots\ldots\ldots\ldots(2)}$$

(1) ÷ (2)

$$\text{R.H.S}=\tan20^\circ\times\frac{3-\tan^2 20^\circ}{\cancel{1-\tan^2 20^\circ}}\times\frac{\cancel{1-\tan^2 20^\circ}}{1-3\tan^2 20^\circ}$$

$$\boxed{\text{R.H.S}=\tan20^\circ\times\frac{3-\tan^2 20^\circ}{1-3\tan^2 20^\circ}=\text{L.H.S}}$$



ii) **Remember that**

In the equation $ax^2+bx+c=0, a \neq 0$

The sum of the roots $= \frac{-b}{a}$

The product of the roots $= \frac{c}{a}$

(The difference of the roots)^2

$= (\text{Sum of the roots})^2 - (4 \times \text{product of the roots})$

$\because$ tan A , tan B are the roots
of the equation $4x^2-17x+15=0$

$$\because \tan(A-B) = \frac{\tan A - \tan B}{1+(\tan A \times \tan B)}$$

$$\boxed{\therefore \tan A + \tan B = \frac{-(-17)}{4} = \frac{17}{4}}$$

$$\boxed{\& \tan A \times \tan B = \frac{15}{4}} \text{..................(1)}$$

$$(\tan A - \tan B)^2 = \left(\frac{17}{4}\right)^2 - \left(4 \times \frac{15}{4}\right) = \frac{49}{16}$$

$$\boxed{\therefore (\tan A - \tan B) = \pm \frac{7}{4}} \text{.................(2)}$$

$$\because \tan(A-B) = \frac{\tan A - \tan B}{1+(\tan A \times \tan B)} \text{..(3)}$$

$\therefore$ From (1) & (2) in (3) we get

$$\tan(A-B) = \frac{\pm \frac{7}{4}}{1+\frac{15}{4}} = \frac{\pm \frac{7}{4}}{\frac{4}{4}+\frac{15}{4}} = \frac{\pm \frac{7}{\cancel{4}}}{\frac{19}{\cancel{4}}} = \pm \frac{7}{19}$$

<u>Another method</u>

$$\because 4x^2-17x+15=0$$

$$\therefore (4x-5)(x-3)=0 \Rightarrow \boxed{x=\frac{5}{4}} \text{ or } \boxed{x=3}$$

Either	**Or**
$\tan A = \frac{5}{4}, \tan B = 3$	$\tan A = 3, \tan B = \frac{5}{4}$
$\therefore \tan(A-B) = \frac{\frac{5}{4}-3}{1+\left(\frac{5}{4} \times 3\right)}$	$\therefore \tan(A-B) = \frac{3-\frac{5}{4}}{1+\left(3 \times \frac{5}{4}\right)}$
$\therefore \tan(A-B) = \frac{-7}{19}$	$\therefore \tan(A-B) = \frac{7}{19}$