Differential calculus and Trigonometry
1st. stage (2nd. Sec.)

Answer all the following questions

1) a) Evaluate:-

(i) $\lim\limits_{x\to 2}\dfrac{(x^2-3)^7-1}{x^2-6x+8}$ (ii) $\lim\limits_{x\to\infty}\left(x-\dfrac{x^2+5}{x-3}\right)$

(iii) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{x\cos x}{(\pi-2x)\sin x}$

b) If $V(t)=t^2-2t+3$ is the relation between the velocity of a body V in cm/sec. and the time t in seconds. Calculate the average rate of change of the velocity of the body during the fifth second and evaluate the rate of change of the velocity at t=4

2)a)
Δ ABC in which a=6 cm , b=10 cm and surface area of Δ ABC=20 cm^2, Given that $\angle ACB$ is an obtuse angle. Find $m(\angle C)$ and the length of $\overline{AB}$.

b)
A fine metal square lamina expands uniformly preserving its shape. Find the variation in its side length when its area varies by 25 cm^2 starting from the instant at which its side length =12 cm.

3)a) If $f(x)=ax^2+bx+4$, Find the variation function v(h) at x=3and if f(3)=4 & $V\left(\dfrac{1}{2}\right)=1\dfrac{3}{4}$. Find a,b.

b) Use the definition to find the first derivative of $f(x)=\sqrt{5-x}$

4)
a) Find the first derivative of each of the following:-

i) $f(x)=2\sqrt{x^3}-\dfrac{2}{x\sqrt{x}}+4\sqrt[3]{x^4}-\dfrac{4\sqrt{x}}{x}+3x^4$

ii) $f(x)=(x\sqrt[3]{x}+1)(x^2\sqrt[3]{x^2}-x\sqrt[3]{x}+1)$

iii) $f(x)=\dfrac{x^2-x+1}{x^2-x-2}$

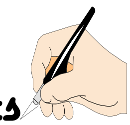
b) In Δ ABC Prove that:-

$\sin A+\sin B+\sin C=\dfrac{4K\Delta}{abc}$

where 2K=a+b+c and Δ is the s.area of the triangle ABC

5)
If $f(x)=\dfrac{2x^2+ax+b}{x^2-5x+4}$ & $f(0)=3, f'(0)=0$.
Find the value of a,b

6)
From the top of a hill the measure of the depression angles of top & base of a building of height = 20 m. were $15^\circ 18'$ and $26^\circ 42'$ respectively. Find the height of the hill.



The answers

1) a)

(i) $\lim\limits_{x\to 2}\dfrac{(x^2-3)^7-1}{x^2-6x+8}$, $f(2)=\dfrac{0}{0}$ (unspecified value)

$$=\lim_{x\to 2}\left(\frac{(x^2-3)^7-1}{(x^2-3)-1}\times\frac{(x^2-3)-1}{x^2-6x+8}\right)$$

$$=\lim_{\substack{x\to 2\\ x^2\to 4\\ x^2-3\to 1}}\frac{(x^2-3)^7-1}{(x^2-3)-1}\times\lim_{x\to 2}\frac{x^2-4}{x^2-6x+8}$$

$$=\lim_{x^2-3\to 1}\frac{(x^2-3)^7-1^7}{(x^2-3)-1}\times\lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-4)}$$

$$=7\times(1)^6\times\frac{4}{-2}=-14$$

(ii)

$$\lim_{x\to\infty}\left(x-\frac{x^2+5}{x-3}\right), f(\infty)=\infty-\infty \text{(unspecified value)}$$

$$=\lim_{x\to\infty}\left(x-\frac{x^2+5}{x-3}\right)=\lim_{x\to\infty}\left(\frac{x(x-3)-(x^2+5)}{x-3}\right)$$

$$=\lim_{x\to\infty}\left(\frac{\cancel{x^2}-3x-\cancel{x^2}-5}{x-3}\right)=\lim_{x\to\infty}\left(\frac{-3x-5}{x-3}\right)$$

Divide up and down by x

$$=\lim_{x\to\infty}\left(\frac{\frac{-3\cancel{x}}{\cancel{x}}-\frac{5}{x}}{\frac{\cancel{x}}{\cancel{x}}-\frac{3}{x}}\right)=\lim_{x\to\infty}\left(\frac{-3-\frac{5}{x}}{1-\frac{3}{x}}\right)$$

$$=\left(\frac{-3-\frac{5}{\infty}}{1-\frac{3}{\infty}}\right)=\left(\frac{-3-0}{1-0}\right)=-3.$$

(iii)

$$\lim_{x\to\frac{\pi}{2}}\frac{x\cos x}{(\pi-2x)\sin x}$$

$$, f\left(\frac{\pi}{2}\right)=\frac{0}{0}\text{(unspecified value)}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{x\cos x}{(\pi-2x)\sin x}$$

$$=\lim_{x\to\frac{\pi}{2}}\left(\frac{x}{\sin x}\times\frac{\cos x}{(\pi-2x)}\right)$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{x}{\sin x}\times\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{(\pi-2x)}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\frac{\pi}{2}}{1}\times\lim_{\substack{x-\frac{\pi}{2}\to 0\\ \left(\frac{\pi}{2}-x\right)\to 0}}\frac{\sin\left(\frac{\pi}{2}-x\right)}{2\left(\frac{\pi}{2}-x\right)}=\frac{\pi}{2}\times\frac{1}{2}=\frac{\pi}{4}$$

1 b-

$v(t) = t^2 - 2t + 3$

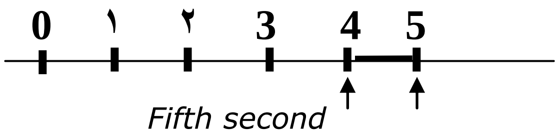
Average rate of change A(h)

$$A(h)=\frac{v(t+h)-v(t)}{h}$$

$$=\frac{(t+h)^2-2(t+h)+3-(t^2-2t+3)}{h}$$

$$=\frac{\cancel{t^2}+2ht+h^2-\cancel{2t}-2h+\cancel{3}-\cancel{t^2}+\cancel{2t}-\cancel{3}}{h}$$

$$=\frac{\cancel{h}(2t+h-2)}{\cancel{h}}=2t+h-2$$



at $t = 4$ and $h = 1$

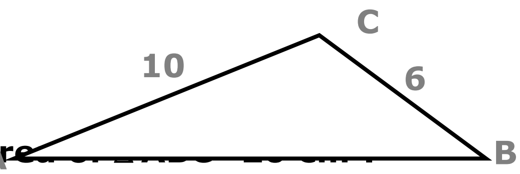
$\therefore t=4 \& h=1 \Rightarrow A(h)=2\times4+1-2=7$ cm/sec

$$\text{Rate of change}=\lim_{h\to 0}\frac{v(t+h)-v(t)}{h}$$

$$=\lim_{h\to 0}(2t+h-2)=2t-2. \ \& \text{ at } t=4$$

$\therefore$ Rate of change$=2\times4-2=6$ cm/sec^2

2 a)



S.area of $\triangle ABC = 20$ cm^2

$$\therefore \frac{1}{2}AC\times BC\sin C=20$$

$$\therefore \frac{1}{2}\times10\times6 \quad \sin C=20 \Rightarrow \sin C=\frac{2}{3}$$

$\therefore m(\angle C)=41^\circ48`$ (refused)

or $m(\angle C)=180-41^\circ48`=138^\circ12`$

$$(AB)^2=(AC)^2+(BC)^2-2\,AC\times BC\cos C$$

$$=(10)^2+(6)^2-2\times10\times6\cos 138^\circ12`$$

$$=136+89.457\approx 225.457$$

$$AB=\sqrt{225.457}\approx 15.15 \text{ Cm}$$

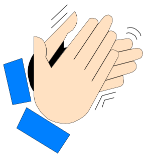
2 b) let the side length = x and the variation of the side length = h

The area of the lamina = x^2

The area of the lamina at the instant of starting variation = 12^2 =144cm²,
The area at the end of variation

$= (12+h)^2 = 144 + 24h + h^2$

$v(h) = \cancel{144} + 24h + h^2 - \cancel{144}$



$\therefore 25 = 24h + h^2 \Rightarrow h^2 + 24h - 25 = 0$

$\therefore (h-1)(h+25) = 0 \Rightarrow$

$\therefore h = 1$

or $h = -25$ **(refused because the lamina expand)**

3 a)

$f(x) = ax^2 + bx + 4 \ , x: 3 \to 3+h$

$v(h) = f(3+h) - f(3)$

$= a(3+h)^2 + b(3+h) + 4 - (9a + 3b + 4)$

$= \cancel{9a} + 6ah + ah^2 + \cancel{3b} + bh + 4 - \cancel{9a} - \cancel{3b} - 4$

$= h(6a + ah + b)$, at $h = \frac{1}{2}$

$\therefore v\left(\frac{1}{2}\right) = \frac{1}{2}\left(6a + \frac{1}{2}a + b\right) = 1\frac{3}{4} = \frac{7}{4}$

$\therefore \frac{1}{\cancel{2}}\left(6a + \frac{1}{2}a + b\right) = \frac{7}{\cancel{2} \times 2}$

$\therefore 6a + \frac{1}{2}a + b = \frac{7}{2}$ by 2

$\therefore 12a + a + 2b = 7 \Rightarrow \boxed{13a + 2b = 7} \quad \boxed{1}$

$f(3) = 4 \Rightarrow 9a + 3b + 4 = 4 \Rightarrow \boxed{3a + b = 0} \ (2)$

from $\boxed{2}$ $b = -3a$ **in** $\boxed{1}$ $13a + 2(-3a) = 7$

$\therefore 7a = 7 \Rightarrow a = 1 \Rightarrow b = -3$

3 b)

$f(x) = \sqrt{5-x} \quad , \forall x \in \left]-\infty, 5\right]$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\sqrt{5-(x+h)} - \sqrt{5-x}}{h}$

$= \lim_{h \to 0} \frac{\sqrt{5-x-h} - \sqrt{5-x}}{h} \times \frac{\sqrt{5-x-h} + \sqrt{5-x}}{\sqrt{5-x-h} + \sqrt{5-x}}$

$= \lim_{h \to 0} \frac{\cancel{5} - \cancel{x} - h - \cancel{5} + \cancel{x}}{h(\sqrt{5-x-h} + \sqrt{5-x})}$

$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(\sqrt{5-x-h} + \sqrt{5-x}} = \frac{-1}{\sqrt{5-x} + \sqrt{5-x}}$

$f'(x) = \frac{-1}{2\sqrt{5-x}} = \frac{-\sqrt{5-x}}{2(5-x)} \quad \forall x \in \left]-\infty, 5\right[$

4 a) i-

$f(x) = 2\sqrt{x^3} - \frac{2}{x\sqrt{x}} + 4\sqrt[3]{x^4} - \frac{4\sqrt{x}}{x} + 3x^4$

$= 2x^{\frac{3}{2}} - 2x^{-\frac{3}{2}} + 4x^{\frac{4}{3}} - 4x^{-\frac{1}{2}} + 3x^4$

$f'(x) = 3x^{\frac{1}{2}} + 3x^{-\frac{5}{2}} + \frac{16}{3}x^{\frac{1}{3}} + 2x^{-\frac{3}{2}} + 12x^3$

$\forall \ x > 0$

ii) $f(x) = (x\sqrt[3]{x} + 1)(x^2\sqrt[3]{x^2} - x\sqrt[3]{x} + 1)$

$= ((x\sqrt[3]{x})^3 + (1)^3) = (x^4 + 1)$

$\therefore f'(x) = 4x^3$

iii)

$f(x) = \frac{x^2 - x + 1}{x^2 - x - 2}$

$f'(x) = \frac{(x^2-x-2)(2x-1) - (x^2-x+1)(2x-1)}{(x^2-x-2)^2}$

$= \frac{-6x+3}{(x^2-x-2)^2} = \frac{-3(2x-1)}{[(x-2)(x+1)]^2}$

$\forall \ x \in R - \{-1, 2\}$

4b)

$$\because \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{C}{\text{Sin } C}$$

$$\because \frac{a+b+c}{\sin A+\sin B+\sin C}=\text{each ratio}$$

$$\therefore \frac{a+b+c}{\sin A+\sin B+\sin C}=\frac{a}{\sin A}$$

$$\therefore a(\sin A+\sin B+\sin C)=\sin A\times 2k$$

$$\sin A+\sin B+\sin C=\frac{\sin A\times 2k}{a}$$

$$=\frac{\frac{1}{2}\,bc\sin A\times 2k}{\frac{1}{2}\,bca}\Rightarrow\frac{\Delta\times 2k}{\frac{1}{2}abc}=\frac{4k\Delta}{abc}$$

Another method

$$\because \text{R.H.S}=\frac{4k\Delta}{abc}$$

$$\therefore \text{R.H.S}=\frac{4\times\frac{1}{2}(a+b+c)\times\frac{1}{2}\cancel{ab}\sin C}{\cancel{ab}c}$$

$$=\frac{(a+b+c)\sin C}{c}=\left(\frac{a}{c}+\frac{b}{c}+1\right)\sin C$$

$$\because \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\therefore \frac{a}{c}=\frac{\sin A}{\sin C}\ \&\ \frac{b}{c}=\frac{\sin B}{\sin c}$$

$$\therefore \text{R.H.S}=\left(\frac{\sin A}{\sin C}+\frac{\sin B}{\sin C}+1\right)\sin C$$

$$=\sin A+\sin B+\sin C=\text{L.H.S}$$

5)

$$f(x)=\frac{2x^2+ax+b}{x^2-5x+4}$$

$$f'(x)=\frac{(x^2-5x+4)(4x+a)-(2x^2+ax+b)(2x-5)}{(x^2-5x+4)^2}$$

$$f'(0)=0$$

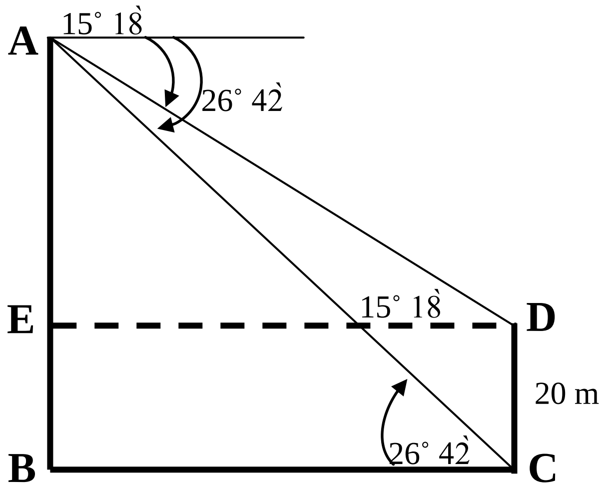
$$\Rightarrow\frac{4xa-b(-5)}{16}=0\Rightarrow\boxed{4a+5b=0}\ (1)$$

$$f(0)=3\Rightarrow\frac{b}{4}=3\Rightarrow\boxed{b=12}\ (2)$$

from (2) in (1) $\therefore 4a+5x12=0$

$4a=-5x12 \quad \Rightarrow a=-5x3=-15$

6)



in Δ ACD

$$m(\angle CAD)=26^\circ 42'-15^\circ 18'=11^\circ 24'$$

$$m(\angle ADC)=90^\circ+15^\circ 18'=105^\circ 18'$$

$$\frac{AC}{\sin(\angle ADC)}=\frac{CD}{\sin(\angle CAD)}$$

$$\frac{AC}{\sin 105^\circ 18'}=\frac{20}{\sin 11^\circ 24'}$$

$$AC=\frac{20\sin 105^\circ 18'}{\sin 11^\circ 24'}\approx 97.599\text{ cm.}$$

in Δ ABC $\quad\frac{AB}{\sin 26^\circ 42'}=\frac{AC}{\sin 90}$

$$\therefore AB=\frac{AC\sin 26^\circ 42'}{\sin 90}$$

$$=\frac{97.599\times\sin 26^\circ 42'}{1}\approx 44\text{ cm.}$$