Algebra First Stage (2nd sec)

Answer all the fallowing questions:

1)a) All terms of an arithmetic sequence are +ve. integers , $T_3+T_7 = 40$ and $T_2^2 + T_5^2 = 521$ Find the sequence and the order of the term whose value equals 83

b) A.S in which $(T_n)=(93 , 89 ,85 ,\ldots\ldots)$
Find: i) The sum of terms starting with T_9 and ending with T_{17} .
ii) Order and value of the first -ve. term and the greatest sum of this sequence.

2)a) ABCD is a quadrilateral in which $m(\angle A)$, $m(\angle B)$, $m(\angle C)$& $m(\angle D)$ form an arithmetic sequence and sin A+ sin D =1
Find the measures of the angles of this quadrilateral.

b) Find the number of terms which should be taken from the sequence (T_n) where:

$$T_n=\begin{cases} 2n+5 & : n \text{ is an odd} \\ 7-2n & : n \text{ is an even} \end{cases}$$

such that their sum equals 147

3)a) Insert n arithmetic means between 75 and 19 given that the ratio between the sum of third and fourth means to the sum of the two means of order n-3 , n-4 is 61 : 37. Find the value of n.

b) (2, b , c , ……, k , m , 512) is a Geometric sequence , if m is 252 more than b. Find the sequence and the number of its terms .

4) a) In a G.S in which its fourth term is 4 and its last term is 64 , if the ratio between the sum of its terms to the sum of their reciprocals is 32 : 1.
Find the sequence & the number of its terms; is this sequence can be sum up to ∞?

b) Prove that:

$X+2X^2+3x^3+4x^4+....$ n terms

$$=\frac{nX^{n+1}}{X-1}-\frac{X(X^n-1)}{(X-1)^2}$$

5) a) The Geometric mean of two numbers and their arithmetic mean and the difference between the two numbers are three consecutive terms of an A.S whose difference is 4 . Find the two numbers and the A.S then find the sum of the first 10 terms of it.

b) If 2X, 3Y, 4Z, 6L are +ve. and form G.S prove that

i) $8YZ<(X+2Z)(Y+2L)$

ii) $\frac{X+2Z}{6Z-X}>\frac{Y}{2L}$

6) The sum of the first three terms in a G.S is 70. If the first term multiply by 4 , the second multiply by 5 and the third multiply by 4 the product form A.S Prove that there are two G.S one of theme can be sum up ∞ . Also find this sum.

b) An A.S is composed of 33 terms the sum of its first eleven terms equals 264; the sum of its last eleven terms equals 330. Find the sum of this sequence, and then find its five middle terms

c) Find the value of the number $i0.2\overline{37}$ in the from of rational number.

The answers

1) a) $T_3+T_7=40 \Rightarrow a+2d+a+6d=40$

$\therefore 2a+8d=40 \Rightarrow a+4d=20 \quad \boxed{1}$

$T_2^2+T_5^2=521 \Rightarrow (a+d)^2+(a+4d)^2=521$

$\therefore (a+d)^2+(20)^2=521 \Rightarrow (a+d)^2=521-400$

$\therefore (a+d)^2=121 \Rightarrow (a+d)^2=(\pm 11)^2$

$\therefore a+d=\pm 11$

$\Rightarrow \boxed{a+d=11}........\boxed{2}$ **(All terms are +ve.)**

$\boxed{1} - \boxed{2} \Rightarrow 3d=9 \Rightarrow \boxed{d=3}$ in $\boxed{1}$

$a=20-4d \Rightarrow a=20-4\times 3=8$

The sequence in (8, 11, 14, ...)

$T_n=83 \Rightarrow a+(n-1)d=83$

$\therefore 8+(n-1)\times 3=83 \Rightarrow 8+3n-3=83$

$\therefore 3n=78 \Rightarrow n=26 \Rightarrow T_{26}=83$

b) $(T_n)=(93, 89, 85, ..)$ is an A .S

$a=93$, $d=89-93=85-89=-4$

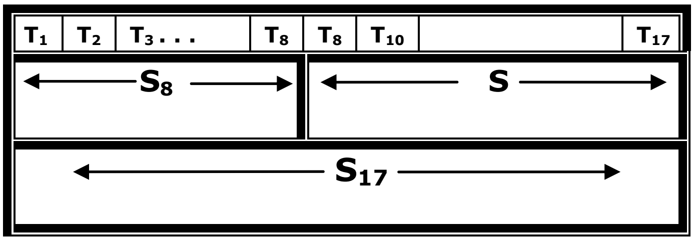
i) $T_9=a+8d=93+8\times -4=61$

$T_{17}=a+16d=93+16\times -4=29$

Number of terms start with T_9 end with T_{17} equals $17-9+1=9$

$S_n=\frac{n}{2}(a+L)=\frac{n}{2}(T_9+T_{17})=\frac{9}{2}(61+29)=405$

Another method



$\because S_n=\frac{n}{2}\left(2a+(n-1)d\right)$ & $\boxed{S=S_{17}-S_8}$

$\therefore S=\frac{17}{2}(2\times 93+16\times -4)-\frac{8}{2}(2\times 93+7\times -4)$

$=1037-632=405$

ii) The –ve.terms $\Longrightarrow T_n<0$

$\therefore a+(n-1)d<0 \Rightarrow 93+(n-1)\times -4<0$

$93-4n+4<0 \Rightarrow 97<4n \Rightarrow 24\frac{1}{4}<n$

$\therefore T_{25}$ is the first -ve term

The greatest sum = the sum of the first 24 terms

$$S_n=\frac{n}{2}\left[2a+(n-1)d\right]=\frac{24}{\cancel{2}}\left[\cancel{2}x93+23x-\overset{2}{\cancel{4}}\right]$$

$$=24(93-46)=1128$$

2)a) let the measure of the angles of the quadrilateral are

A	B	C	D
x-3y	x-y	x+y	x+3y

The sum of measures of A,B,C and D equal 360°

$\therefore x-\cancel{3y}+x-\cancel{y}+x+\cancel{y}+x+\cancel{3y}=360^\circ$

$\therefore 4x=360^\circ \Rightarrow x=90^\circ$

$\therefore m(\angle A)=90^\circ-3y \ \ \& m(\angle D)=90^\circ+3y$

$\because \sin A+\sin D=1$

$\therefore \sin(90^\circ-3y)+\sin(90^\circ+3y)=1$

$\therefore \cos 3y+\cos 3y=1 \Rightarrow 2\cos 3y=1$

$\therefore \cos 3y=\frac{1}{2} \Rightarrow \boxed{\cos 3y=\cos 60^\circ}$ or $\boxed{\cos 3y=\cos 300^\circ}$

Either	Or
$3y=60^\circ$	$3y=300^\circ$
$\therefore \boxed{y=20^\circ}$	$\boxed{y=100^\circ}$
$\therefore m(\angle A)=30^\circ$	$\therefore m(\angle A)=-210^\circ$
$m(\angle B)=70^\circ$	–ve. Measure
$m(\angle C)=110^\circ$	Refused
$m(\angle D)=150^\circ$	

b)

$$(T_n)=\begin{cases}2n+5 & :\text{n is an odd number}\\ 7-2n & :\text{n is an even number}\end{cases}$$

$T_1=2+5=7$ $\qquad T_2=7-4=3$

$T_3=6+5=11$ $\qquad T_4=7-8=-1$

$T_5=10+5=15$ $\qquad T_6=7-12=-5$

The sequence (7, 3, 11,-1, 15,-5, ...)

$\because S_n=147$, **n is either even number or odd number**

At n even : then the number of of terms of odd order = the number of terms of even order = $\frac{n}{2}$

The odd order terms (7,11,15 ,........)

A.S a = 7, d=4, number of terms $\frac{n}{2}$

$\because S_n = n\left[2a+(n-1)d\right]$

$$\therefore S'_{\frac{n}{2}} = \frac{\frac{n}{2}}{2}\left[2\times 7+(\frac{n}{2}-1)\times 4\right]$$

$$= \frac{n}{4}(14+2n-4) = \frac{n}{\cancel{4}^{2}}\left(\cancel{10}^{5}+\cancel{2n}^{n}\right) = \frac{n}{2}(5+n)$$

The even order terms (3,-1,-5, ...)

A.S a=3,d=-4 & number of terms $\frac{n}{2}$

$$S''_{\frac{n}{2}} = \frac{\frac{n}{2}}{2}\left[2\times 3+\left(\frac{n}{2}-1\right)\times -4\right] = \frac{n}{4}(6-2n+4)$$

$$= \frac{n}{4}(10-2n) = \frac{n}{2}(5-n)$$

$$S_n = S'_{\frac{n}{2}} + S''_{\frac{n}{2}} = 147$$

$$\therefore \frac{n}{2}(5+n)+\frac{n}{2}(5-n) = 147 \qquad \times 2$$

$$\therefore 5n+\cancel{n^2}+5n-\cancel{n^2} = 294$$

$10n=294 \Rightarrow n=29.4 \notin Z^+$. **Refused**

At n odd $\Rightarrow$ **The number of terms of odd order exceed one to the number of terms of even order**

$\therefore$ **Number of odd order terms =** $\frac{n-1}{2}+1$

$\therefore$ **Number of even order terms =** $\frac{n-1}{2}$

$$S'_{\frac{n-1}{2}} = \frac{\frac{n-1}{2}+1}{\cancel{2}}\left[\cancel{2}^{1}\times 7+(\frac{n-1}{2}+\cancel{1}-\cancel{1})\times \cancel{4}^{2}\right]$$

$$=(\frac{n-1}{2}+\frac{2}{2})(7+n-1) = (\frac{n+1}{2})(6+n)$$

$$S''_{\frac{n-1}{2}} = \frac{\frac{n-1}{2}}{\cancel{2}}\left[\cancel{2}^{1}\times 3+\left(\frac{n-1}{2}-1\right)\times -\cancel{4}^{2}\right]$$

$$= \frac{n-1}{2}(3-n+1+2) = \frac{n-1}{2}(6-n)$$

$$S_n = S'_{\frac{n-1}{2}+1} + S''_{\frac{n-1}{2}} = 147$$

$$\therefore \frac{n+1}{2}(6+n)+\frac{n-1}{2}(6-n) = 147 \qquad \times 2$$

$$(n+1)(6+n)+(n-1)(6-n) = 294$$

$$6n+\cancel{n^2}+6+n+6n-\cancel{n^2}-6+n = 294$$

$$14n = 294 \Rightarrow n = 21 \in Z^+$$

$\therefore S_{21}$ **of this sequence = 147**

3)a)

	M_1	M_2		M_{n-1}	M_n	
75,	75+d,	75+2d,	. . .	19-2d	,19-d	,19
T_1	n A. means					T_{n+2}
A.S						

$$\frac{M_3+M_4}{M_{n-3}+M_{n-4}}=\frac{61}{37}\Rightarrow\frac{T_4+T_5}{T_{n-2}+T_{n-3}}=\frac{61}{37}$$

$$\therefore\frac{(75+3d)+(75+4d)}{(19-4d)+(19-5d)}=\frac{61}{37}\Rightarrow\frac{150+7d}{38-9d}=\frac{61}{37}$$

$$38\times61-9\times61d=150\times37+7\times37d$$

$$2318-549d=5550+259d$$

$$-808d=3232\Rightarrow\boxed{d=-4}$$

$$T_{n+2}=19\Rightarrow a+(n+2-1)d=19$$

$$\therefore 75+(n+1)\times-4=19\Rightarrow 75-4n-4=19$$

$$\therefore 4n=52\Rightarrow\boxed{n=13}$$

b) $m=b+252$ $\boxed{1}$

$$\frac{b}{2}=\frac{512}{m}\Rightarrow mb=1024 \quad \boxed{2}$$

from (1) in (2) $b(b+252)=1024$

$$\therefore b^2+252b-1024=0\Rightarrow(b-4)(b+256)=0$$

$b-4=0\Rightarrow b=4$	Or $b+256=0\Rightarrow b=-256$
in (1) $m=256$	in (1) $m=-4$
$\therefore r=2$	$\therefore r=-128$. Refused

The sequence: $(2,4,8,\ldots\ldots.512)$

$$T_n=ar^{n-1}\Rightarrow 2(2)^{n-1}=512$$

$$2^{n-1}=256\Rightarrow 2^{n-1}=2^8\Rightarrow n-1=8\Rightarrow n=9$$

4) a) $T_4 = 4 \Rightarrow ar^3 = 4 \qquad (1)$

$T_n = 64 \Rightarrow ar^{n-1} = 64 \qquad (2)$

$a, ar, ar^2 \ldots\ldots \Rightarrow S_n = \dfrac{a(r^n-1)}{r-1}$

$\dfrac{1}{a}, \dfrac{1}{ar}, \dfrac{1}{ar^2}, \ldots\ldots \Rightarrow S'_n = \dfrac{\frac{1}{a}\left[1-\left(\frac{1}{r}\right)^n\right]}{1-\frac{1}{r}}$

$\therefore S'_n = \dfrac{\frac{1}{a}\left(\frac{r^n}{r^n}-\frac{1}{r^n}\right)}{\frac{r-1}{r}} = \dfrac{1}{a} \times \dfrac{r^n-1}{r^n} \times \dfrac{r}{r-1} = \dfrac{r(r^n-1)}{ar^n(r-1)}$

$\because \dfrac{S_n}{S'_n} = \dfrac{32}{1} \Rightarrow \dfrac{a\cancel{(r^n-1)}}{\cancel{r-1}} \times \dfrac{ar^n\cancel{(r-1)}}{r\cancel{(r^n-1)}} = \dfrac{32}{1}$

$\therefore a^2r^{n-1} = 32 \quad (3) \qquad$ from (2) in (3)

$a \times ar^{n-1} = 36 \Rightarrow a \times 64 = 32 \Rightarrow \boxed{a = \frac{1}{2}}$

in (1) $\Rightarrow \dfrac{1}{2}r^3 = 4 \Rightarrow r^3 = 8 \Rightarrow r = 2$

in (2) $\dfrac{1}{2}(2)^{n-1} = 64 \Rightarrow (2)^{n-1} = 128$

$(2)^{n-1} = 2^7 \Rightarrow n-1 = 7 \Rightarrow \boxed{n=8}$

b) $S = X + 2X^2 + 3X^3 + 4X^4 + \ldots\ldots + nX^n \quad (1)$

Multiply by X

$\therefore S\,X = X^2 + 2X^3 + 3X^4 + 4X^5 + \ldots\ldots + nX^{n+1} \qquad (2)$

(2)-(1)

$S\,X - S = (-X - X^2 - X^3 - X^4 \ldots\ldots - X^n) + nX^{n+1}$

$S(X-1) = -(X + X^2 + X^3 + X^4 \ldots\ldots + X^n) + nX^{n+1}$

$= \left(\text{Sum of G.S in which } a=x, r=x, L=x^n\right) + nx^{n+1}$

$S(X-1) = \dfrac{Lr-a}{r-1} + nX^{n+1} = -\left(\dfrac{X^n \times X - X}{X-1}\right) + nX^{n+1}$

$= -\dfrac{X(X^n-1)}{X-1} + nX^{n+1} \Rightarrow \boxed{S = -\dfrac{X(X^n-1)X}{(X-1)^2} + \dfrac{nX^{n+1}}{X-1}}$

5) a) Let the two numbers are a&b

$\therefore \sqrt{ab}\ , \frac{a+b}{2}, a-b$ is an A.S

$\therefore \cancel{2}(\frac{a+b}{\cancel{2}})=\sqrt{ab}+(a-b)$

$\cancel{a}+b=\sqrt{ab}+\cancel{a}-b \Rightarrow 2b=\sqrt{ab}$

By squaring both sides

$4b^2=ab \Rightarrow 4b^2-ab=0 \Rightarrow b(4b-a)=0$

Either $4b-a=0 \Rightarrow \boxed{a=4b\ (1)}$, or $b=0$ refused.

The difference $=4 \Rightarrow a-b-\frac{a+b}{2}=4 \quad \times 2$

$\therefore 2a-2b-a-b=8 \Rightarrow a-3b=8 \quad (2)$

From (1) in (2)

$4b-3b=8 \Rightarrow b=8$ in (1)

$a=4\times 8=32 \Rightarrow$ The two number are 8,32

The sequence $(\sqrt{8\times 32}, \frac{8+32}{2}, 32-8, \ldots)$

The sequence is (16, 20, 24…)

$S_{10}=\frac{10}{2}[2\times 16+9\times 4]=5[32+36]=340$

b)i) 2X,3Y ,4Z is G .S

3Y is a G.mean between 2X,4Z

$\therefore$ A.mean of 2X , 4Z is $\frac{2x+4Z}{2}=X+2Z$

$\because$ G mean < A.mean $\Rightarrow 3Y < x+2Z \quad (1)$

3Y,4Z,6L is G.S

$\therefore$ 4Z is the G.meanbetween 3Y,6L

A mean of 3Y ,6L is $\frac{3Y+6L}{2}=\frac{3(Y+2L)}{2}$

$\because$ G.mean < A.mean

$\therefore 4Z<\frac{3(y+2L)}{2}$ . . . . (2)

(1) x (2)

$\cancel{3}Y\times 4Z<(x+2Z)\times\frac{\cancel{3}(y+2L)}{2} \qquad \times 2$

$8yz<(x+2z)(y+2L)$

ii) $8YZ< XY+2XL+2YZ+4ZL$

$6YZ-XY < 2XL+4ZL$

$Y(6Z-X) < 2L(X+2Z)$

$\boxed{\frac{y}{2L}<\frac{x+2Z}{6z-x}}$

6) a) $(T_n) = (a, ar, ar^2, ...)$ G.S

$a + ar + ar^2 = 70 \Rightarrow a(1 + r + r^2) = 70 \quad (1)$

$(4a, 5ar, 4ar^2, ...)$ form A.S

$\therefore 2 \times 5ar = 4a + 4ar^2 \Rightarrow \cancel{2} \times 5\cancel{a}r = \overset{2}{\cancel{4}}\,\cancel{a}(1 + r^2)$

$5r = 2 + 2r^2 \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (2r - 1)(r - 2) = 0$

$2r - 1 = 0 \Rightarrow \boxed{r = \frac{1}{2}}$

in(1)

$a\,(1 + \frac{1}{2} + \frac{1}{4}) = 70$

$a \times \left(\frac{4+2+1}{4}\right) = 70$

$a \times \frac{7}{4} = 70 \Rightarrow a = 40$

The sequence

(40, 20, 10, ...)

$\because |r| < 1 \Rightarrow$ The sequence can be sum up to ∞

$S_\infty = \frac{a}{1-r} = \frac{40}{1 - \frac{1}{2}} = \frac{40}{\frac{1}{2}} = 80$

or $r - 2 = 0 \Rightarrow \boxed{r = 2}$

in(1)

$a(1 + 2 + 4) = 70$

$a \times 7 = 70$

$a = 10$

The sequence

(10, 20, 40,......)

$r > 1 \Rightarrow$ The sequence can\`t be sum up to ∞

b) $\underbrace{T_1, T_2, T_3, ...T_{11}}_{S_{11}\text{ First}}, T_{12}, ..., \underbrace{T_{23}, T_{24}, ...T_{33}}_{S'_{11}\text{ Last}}$

$S_{11} = 264$

$\frac{11}{2}(2a + 10d) = 264$

$a + 5d = 24 ...(1)$

$S'_{11} = 330$

$T_{23} = a + 22d$

$S'_{11} = \frac{11}{\cancel{2}}\left(\cancel{2}T_{23} + \overset{5}{\cancel{10}}d\right) = 330$

$\therefore a + 22d + 5d = 30$

$\therefore a + 27d = 30 \; (2)$

$(2) - (1) \Rightarrow 22d = 6 \Rightarrow \boxed{d = \frac{3}{11}}$

In (1) $a = 24 - 5 \times \frac{3}{11} = \frac{249}{11}$

$S_{33} = \frac{33}{\cancel{2}}\left(\cancel{2}x\frac{249}{11} + \overset{16}{\cancel{32}} \times \frac{3}{11}\right) = 33\left(\frac{249}{11} + \frac{48}{11}\right)$

$= 33 \times \frac{297}{11} = 33 \times 27 = 891$

Since number of terms is an odd, then there is only one middle term which is:-

$$T_{\frac{33+1}{2}} = T_{17} = a + 16d = \frac{249}{11} + 16 \times \frac{3}{11} = \frac{297}{11}$$

The five middle terms are:

$$\frac{291}{11}, \frac{294}{11}, \frac{297}{11}, \frac{300}{11}, \frac{303}{11}$$