

Final Revision for 3rd sec Algebra Solid Geometry

منتدى توجيه الرياضيات

Algebra

✓ Permutations and combinations

- $\underline{n} = {}^n P_n = n(n-1)(n-2) \times \dots \times 3 \times 2 \times 1$
- $\underline{n} = n \underline{n-1}$
- ${}^n P_r = \frac{\underline{n}}{\underline{n-r}}$ ■ ${}^n P_0 = 1$ ■ ${}^n P_1 = n$
- ${}^n C_r = \frac{{}^n P_r}{\underline{r}}$ ■ ${}^n C_r = \frac{\underline{n}}{\underline{r} \underline{n-r}}$ ■ ${}^n C_n = 1$
- ${}^n C_r = {}^n C_{n-r}$
- If ${}^n C_r = {}^n C_d \quad \therefore r = d \quad \text{or} \quad r + d = n$
- $\frac{{}^n C_r}{{}^n C_{r-1}} = \frac{n-r+1}{r}$ ■ ${}^n C_r + {}^n C_{r+1} = {}^{n+1} C_{r+1}$
- ${}^n C_0 + {}^n C_1 + {}^n C_2 + \dots + {}^n C_{n-1} + {}^n C_n = 2^n$

✓ Binomial theorem

$$(a+b)^n = {}^n C_0 a^n + {}^n C_1 b a^{n-1} + {}^n C_2 b^2 a^{n-2} + \dots + {}^n C_r b^r a^{n-r} + \dots + {}^n C_n b^n$$

- ✓ The general term for the expansion of $(a+b)^n$: $T_{r+1} = {}^n C_r b^r a^{n-r}$

✓ Remarks:

- 1) In the expansion $(a+b)^n$ the number of its elements equals $(n+1)$ elements.
- 2) $(1+x)^n = 1 + {}^n C_1 x + {}^n C_2 x^2 + \dots + {}^n C_r x^r + \dots + x^n$.
- 3) $(1-x)^n = 1 - {}^n C_1 x + {}^n C_2 x^2 - \dots + {}^n C_r (-x)^r + \dots + (-x)^n$.
- 4) $(a+b)^n + (a-b)^n = 2(T_1 + T_3 + T_5 + \dots)$
- 5) $(a+b)^n - (a-b)^n = 2(T_2 + T_4 + T_6 + \dots)$

✓ The middle term or the two middle terms in $(a + b)^n$:

i) If n is *even* then the expansion has one mid-term its $T_{\frac{n}{2}+1}$

ii) If n is *odd* then the expansion has two mid-terms they are $T_{\frac{n+1}{2}}, T_{\frac{n+3}{2}}$

✓ In the expansion of $(x + a)^n$, The ratio between two consecutive terms

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times \frac{a}{x}$$

✓ Complex numbers

▪ $i^2 = -1$ So $i^3 = -i$, $i^4 = 1$ so $i^{4n} = 1, n \in \mathbb{Z}$

▪ If $Z = x + iy$ $\therefore \overline{Z} = x - iy$

▪ If $Z = x + yi$ $\therefore |Z| = \sqrt{x^2 + y^2} = r$ (*modulus Z*)

$\tan \theta = \frac{y}{x}$ θ is called the amplitude of Z (*or argument of Z*)

The amplitude is called principle if $\theta \in [-\pi, \pi]$

$Z = r(\cos \theta + i \sin \theta) \rightarrow$ The trig. Form for Z



Trig. Form by using calculator

$$\sqrt{3} - i \rightarrow = \rightarrow \text{shift} \rightarrow 2 \rightarrow 3 \rightarrow =$$

Gives $2 \angle -30$

▪ If $Z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$, $Z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$

$\therefore Z_1 \times Z_2 = r_1 \times r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$

$\therefore \frac{Z_1}{Z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$

$\therefore Z^n = r^n (\cos n\theta + i \sin n\theta)$

$Z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$



alg. form



Trig. form



Exp. form

■ De Moiver's theorem

$$(\cos\theta + i\sin\theta)^{\frac{1}{k}} = \cos\frac{\theta + 2m\pi}{k} + i\sin\frac{\theta + 2m\pi}{k}$$

where $m \in \{0, 1, 2, \dots, k-1\}$

✓ The alg. form the cubic roots of one are $1, \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right), \left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}i\right)$

✓ The cubic roots of one are $1, \omega, \omega^2$

i)	$1 + \omega + \omega^2 = 0$	$1 + \omega = -\omega^2$
		$1 + \omega^2 = -\omega$
		$\omega + \omega^2 = -1$
ii)	$\omega^3 = 1$	$\frac{1}{\omega} = \omega^2$
		$\frac{1}{\omega^2} = \omega$
iii)	$\omega^2 - \omega = \pm\sqrt{3}i$	
iv)	$\omega^{3n} = 1, \omega^{3n+1} = \omega, \omega^{3n+2} = \omega^2, (n \in \mathbb{Z})$	

✓ Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & l & x \\ b & m & y \\ c & n & z \end{vmatrix} = a \begin{vmatrix} m & y \\ n & z \end{vmatrix} - l \begin{vmatrix} b & y \\ c & z \end{vmatrix} + x \begin{vmatrix} b & m \\ c & n \end{vmatrix}$$

The signs of the cofactors: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$



Properties of determinants:

- (1) In any det. If the rows are replaced by the columns and the columns by the rows in the same order, the value of the det. is unchanged. **For example:** $\begin{vmatrix} 1 & 3 & 6 \\ a & b & c \\ x & y & z \end{vmatrix} = \begin{vmatrix} 1 & a & x \\ 3 & b & y \\ 6 & c & z \end{vmatrix}$

- (2) The value of the det. is unchanged by expanding it in terms of the elements of any of its rows (or columns).

- (3) The value of determinant equals Zero in the following cases :

- a) If the corresponding elements in two rows (columns) of any det. are equal, then the value of the det. equals zero.

For example: $\begin{vmatrix} 7 & 6 & 3 \\ 7 & 6 & 3 \\ x & y & z \end{vmatrix} = 0$, and $\begin{vmatrix} 3 & 5 & 5 \\ 7 & 8 & 8 \\ -5 & -1 & -1 \end{vmatrix} = 0$

- b) If all the elements of any row (column) in a det. are zeroes, then the value of the det. is zero. **For example:** $\begin{vmatrix} 0 & 0 & 0 \\ x & y & z \\ 5 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 7 \\ 2 & 0 & 2 \\ 4 & 0 & -1 \end{vmatrix} = 0$

- (4) If there is a common factor in all the elements of any row (column) in a det. then this factor can be taken outside the det.

For example : $\begin{vmatrix} 5 & 10 & 15 \\ 6 & 8 & 3 \\ 8 & 16 & 7 \end{vmatrix} = 5 \begin{vmatrix} 1 & 2 & 3 \\ 6 & 8 & 3 \\ 8 & 16 & 7 \end{vmatrix} = 5 \times 2 \begin{vmatrix} 1 & 1 & 3 \\ 6 & 4 & 3 \\ 8 & 8 & 7 \end{vmatrix}$

- (5) In any det., if the positions of any two rows (columns) are interchanged, then the value of the resulting det. is equal to the value of the original det. multiplied by (-1).

For example: $\begin{vmatrix} 7 & a & l \\ 6 & c & m \\ 2 & d & n \end{vmatrix} = - \begin{vmatrix} a & 7 & l \\ c & 6 & m \\ d & 2 & n \end{vmatrix}$

- (6) In a det., if all elements of any row (column) are written as the sum of two elements, the value of the det. can be written as the sum of two det.

For example : $\begin{vmatrix} 3 & 6 & 2 \\ 5 & -1 & 4 \\ 0 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 7 & 2 \\ 5 & -4 & 4 \\ 0 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 13 & 2 \\ 5 & -5 & 4 \\ 0 & 11 & 2 \end{vmatrix}$

- (7) In any det., if we add to all elements of any row (column) a multiple of the elements of other row (column), then the value of det. is unchanged.

As $\begin{vmatrix} a & x & l \\ b & y & m \\ c & z & n \end{vmatrix} \xrightarrow{3c_3 + c_1} \begin{vmatrix} a+3l & x & l \\ b+3m & y & m \\ c+3n & z & n \end{vmatrix}$ and so on.....

- (8) In any det., if we multiply the elements of any row (column) by the cofactor of the corresponding elements of another row (column) and we sum the result, we get the value zero.

- (9) The value of any det. in the triangle form is equal to the product of the elements of the main diagonal.

For example: $\begin{vmatrix} 6 & 0 & 0 \\ 7 & 5 & 0 \\ 8 & -1 & 3 \end{vmatrix} = 6 \times 5 \times 3 = 90$, $\begin{vmatrix} 8 & -3 & 6 \\ 0 & 7 & 3 \\ 0 & 0 & 2 \end{vmatrix} = 8 \times 7 \times 2 = 112$

Matrices

If $A = \begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}$, then $A^t = \begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix}$, A^t is called **transpose of matrix A** and $(A^t)^t = A$

Note: $(A + B)^t = (B + A)^t$, $(A + B)^t = A^t + B^t$

Multiplication of matrices

If A is a matrix of order $m \times n$ and B is another matrix of order $x \times y$ then we can find $A \times B$ when $n = x$ and the matrix $(A \times B)$ is of order $m \times y$

Note: i) $A \times B \neq B \times A$ ii) $(A \times B)^t = B^t \times A^t$

The multiplicative inverse of a square matrix in order (2×2)

- We studied before, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a square matrix of (2×2) then its multiplicative inverse $A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$
- And we knew that: if $|A| = 0$ then the matrix has no multiplicative inverse and the matrix called **singular matrix**

Remarks :

- 1) The matrix A has a **multiplicative inverse** if $|A| \neq 0$
- 2) If A is a square matrix and A^{-1} is **its mult. Inv.** then $AA^{-1} = A^{-1}A = I$ where I is a **unit matrix**.

The cofactor of a matrix [C]

- If $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$, then the matrix $C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$ where

$$C_{11} = (-1)^{1+1} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix}, C_{12} = (-1)^{1+2} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix}, C_{13} = (-1)^{1+3} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}$$

and so on . Then the matrix C is **the cofactor of the matrix** .

- So we can summaries the signs which connects between the small determinist and the cofactor of any element of any square matrix by $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

The adjoint matrix

If A is a square matrix and C is its cofactor matrix then **the adjoint matrix** of A which denoted by A^{adj} is **the transpose of the matrix C** so: $A^{adj} = C^t$.

The multiplicative inverse of the square matrix of order 3×3 .

- If A is a square matrix in order 3×3 then A^{-1} is called **the multiplicative inverse** of the matrix A . Where $A^{-1} = \frac{1}{|A|} A^{adj}$ where $|A| \neq 0$.
- Any square matrix has $|A| = 0$, so **it has no multiplicative inverse** and A is called **singular matrix**.

Properties of the multiplicative inverse of a matrix

$$\begin{array}{lll} (1) (AB)^{-1} = B^{-1}A^{-1} & (2) (A^{-1})^t = (A^t)^{-1} & \\ (3) (A^{-1})^{-1} = A & (4) (A^{-1})^2 = (A^2)^{-1} & (5) (I)^{-1} = I \end{array}$$

The rank of the matrix [RK]

- The rank of a non-zero matrix is the greatest order of determinant or a small determinant of the matrix that its value not equals zero.
- As if A is a non-zero matrix in order $m \times n$ where $m \geq n$, then the rank of the matrix A denoted by $RK(A)$ where $1 \leq RK(A) \leq n$.

Remarks

- i. If (I) is a unit matrix in order $(m \times m)$, then the rank of $(I) = m$, because $|I| = 1 \neq 0$.
- ii. The rank of zero matrix equals zero.

Augmented matrix (A^*)

If we have m of linear equations in n unknowns and we write them in the form $AX = B$, then the augmented matrix (A^*) is defined as : $(A^*) = (A|B)$ and its order is $(m \times (n + 1))$

Possibility of solving a system of linear equations

■ First: The non-homogenous equations

The system of equations $AX = d$ is called non-homogenous when $d \neq \square$ and

For a system of (n) non-homogenous equations and in (n) unknowns:

- i) If $RK(A) = RK(A^*) = n$, then the system has a unique solution .
- ii) If $RK(A) = RK(A^*) = k$, $k < n$ then the system has infinite solutions ,
- iii) If $RK(A) \neq RK(A^*)$, then the system has no solution.

The following table shows the existence of the solution of a system of three equations in three unknowns:

$RK(A^*)$	$RK(A)$	<i>Solution existence</i>
3	3	<i>unique solution</i>
3	2	<i>No solution</i>
3	1	<i>No solution</i>
2	2	<i>Infinite set of solutions</i>
2	1	<i>No solution</i>
1	1	<i>Infinite set of solutions</i>

■ Second: The homogenous equations

The system of homogenous equations is written by $AX = \square$ and the rank of the augmented matrix A^* equals the rank of the matrix A so $RK(A) = RK(A^*)$ and in this case :

- If the number of unknowns $n = RK(A) = RK(A^*)$ so this system has a unique solution **it is the zero solution.**
- If $RK(A) < n$ then the system has infinite set of solutions including the zero solution.

Analytic solid Geometry

✓ Cartesian planes

The Cartesian plane xy its equation is $Z = 0$

The Cartesian plane xz its equation is $y = 0$

The Cartesian plane yz its equation is $x = 0$

✓ The distance between two points

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ are two points in space, then the length of the line segment \overline{AB} is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

✓ The coordinates of the midpoint of a line segment

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ are two points in space, then the coordinates of the midpoint of \overline{AB} are: $C = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$

📖 The equation of the sphere in space

- The equation of sphere whose center is (l, k, n) and its radius length is r is $(x-l)^2 + (y-k)^2 + (z-n)^2 = r^2$
- The equation of the sphere whose center is the origin and radius length r is $x^2 + y^2 + z^2 = r^2$
- The equation of the sphere: $x^2 + y^2 + z^2 + 2Lx + 2ky + 2nz + d = 0$ where its center is $(-l, -k, -n)$ and length of its radius length is $r = \sqrt{l^2 + k^2 + n^2 - d}$ where $l^2 + k^2 + n^2 > d$

📖 Vectors

✓ The norm of a vector

If $\vec{A} = (A_x, A_y, A_z)$ then $\|\vec{A}\| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

✓ Equality of vectors in space

If $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$ then $\vec{A} = \vec{B}$ if and only if:

$$A_x = B_x, A_y = B_y, A_z = B_z$$

✓ The unit vector

The unit vector is a vector whose norm equals the unit length.

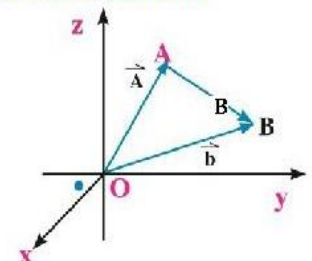
$\hat{i} = (1, 0, 0)$ The unit vector in the +ve direction of x - axis

$\hat{j} = (0, 1, 0)$ The unit vector in the +ve direction of y - axis

$\hat{k} = (0, 0, 1)$ The unit vector in the +ve direction of z - axis

✓ Expressing a directed line segment in the space in terms of the coordinates of its terminals

If A and B are two points in space their position vectors are \vec{A} and \vec{B} respectively, then $\overrightarrow{AB} = \vec{B} - \vec{A}$



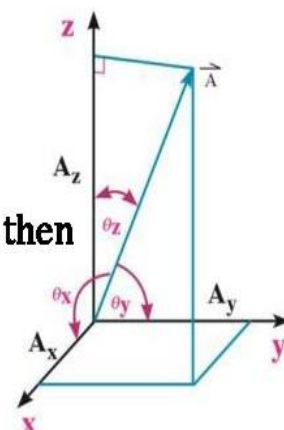
✓ The unit vector in the direction of a given vector

If $\vec{A} = (A_x, A_y, A_z) \in R^3$, then the vector \vec{U}_A is called the unit vector in the direction of \vec{A} $\vec{U}_A = \frac{\vec{A}}{\|\vec{A}\|}$

✓ Direction angles and direction cosines of a vector in space

If $(\theta_x, \theta_y, \theta_z)$ are the measures of the angles made by the vector

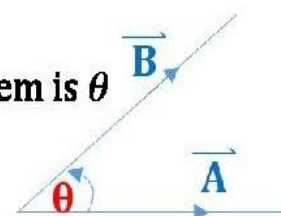
$\vec{A} = (A_x, A_y, A_z)$ and with the +ve directions of x, y, z axes respectively, then



- $A_x = \|\vec{A}\| \cos \theta_x$, $A_y = \|\vec{A}\| \cos \theta_y$, $A_z = \|\vec{A}\| \cos \theta_z$
- $(\theta_x, \theta_y, \theta_z)$ are called the direction angles of vector \vec{A}
- $\cos \theta_x, \cos \theta_y, \cos \theta_z$ are called the direction cosines of vector \vec{A}
- $\cos \theta_x \hat{i} + \cos \theta_y \hat{j} + \cos \theta_z \hat{k}$ represents the unit vector in the direction of \vec{A}
- $\cos^2 \theta_x + \cos^2 \theta_y + \cos^2 \theta_z = 1$

✓ The scalar product of two vectors

If \vec{A} and \vec{B} are two vectors in R^3 and the measure of the angle between them is θ where $0 \leq \theta \leq 180^\circ$, then $\vec{A} \cdot \vec{B} = \|\vec{A}\| \|\vec{B}\| \cos \theta$



✓ Properties of the scalar products:

- 1) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- 2) $(\vec{A} + \vec{B}) \cdot \vec{C} = \vec{A} \cdot \vec{C} + \vec{B} \cdot \vec{C}$
- 3) If $K \in R \therefore K \vec{A} \cdot \vec{B} = \vec{A} \cdot K \vec{B} = K(\vec{A} \cdot \vec{B})$
- 4) $\vec{A} \cdot \vec{A} = \|\vec{A}\|^2$
- 5) If $\vec{A} \perp \vec{B} \therefore \vec{A} \cdot \vec{B} = 0$ and if $\vec{A} \cdot \vec{B} = 0$

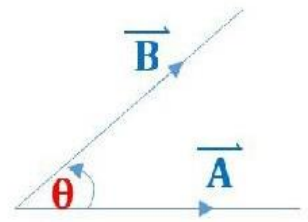
✓ The scalar product of two vectors in an orthogonal coordinate system

If $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$

$$\therefore \vec{A} \cdot \vec{B} = (A_x, A_y, A_z) \cdot (B_x, B_y, B_z) = A_x B_x + A_y B_y + A_z B_z$$

✓ The angle between two vectors: $\cos \theta = \frac{\vec{A} \cdot \vec{B}}{\|\vec{A}\| \|\vec{B}\|}$

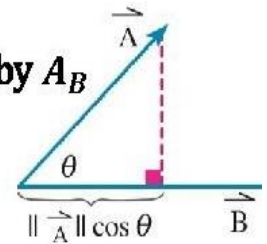
- i) If $\cos \theta = 1$, then $\vec{A} // \vec{B}$ and on the same direction
- ii) If $\cos \theta = -1$, then $\vec{A} // \vec{B}$ and on the opposite direction
- iii) If $\cos \theta = 0$, then $\vec{A} \perp \vec{B}$



✓ The component of a vector in the direction of another vector

The component of the vector \vec{A} in the direction of vector \vec{B} is denoted by A_B

$$A_B = \|\vec{A}\| \cos \theta = \frac{\vec{A} \cdot \vec{B}}{\|\vec{B}\|}$$



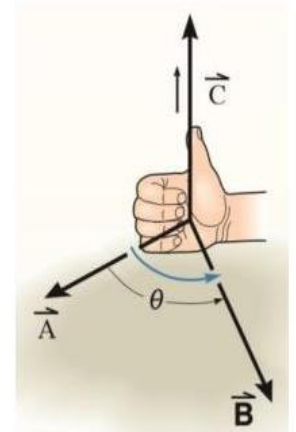
✓ The work done by the force \vec{F} to make a displacement \vec{S}

$$\text{The work } (W) = \vec{F} \cdot \vec{S} = \|\vec{F}\| \|\vec{S}\| \cos \theta$$

- i) If the force \vec{F} is in the direction of the displacement ($\theta = 0^\circ$), then $W = \|\vec{F}\| \|\vec{S}\|$
- ii) If the force \vec{F} is in the opposite direction of the displacement ($\theta = 180^\circ$), then $W = -\|\vec{F}\| \|\vec{S}\|$
- iii) If the force \vec{F} is perpendicular to the direction of the displacement ($\theta = 90^\circ$), then $W = 0$

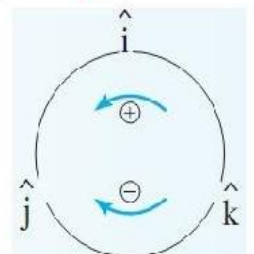
✓ The vector product of two vectors

If \vec{A} and \vec{B} are vectors in R^3 and the measure of the smallest angle between them is θ Then $\vec{A} \times \vec{B} = (\|\vec{A}\| \|\vec{B}\| \sin \theta) \vec{C}$ where \vec{C} is a unit vector perpendicular to the plane which contains \vec{A} and \vec{B} .



✓ The properties of the vector product of two vectors

- 1- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- 2- $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
- 3- If $\vec{A} \times \vec{B} = \vec{0}$, then $\vec{A} // \vec{B}$ or one of the two vectors or both of them equals $\vec{0}$
- 4- $\hat{i} \times \hat{j} = \hat{k}$ and $\hat{j} \times \hat{i} = -\hat{k}$
 $\hat{j} \times \hat{k} = \hat{i}$ $\hat{k} \times \hat{j} = -\hat{i}$
 $\hat{k} \times \hat{i} = \hat{j}$ $\hat{i} \times \hat{k} = -\hat{j}$



✓ The cross production in Cartesian coordinates

■ If $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$ two vectors, then $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

■ **Special case:** If $\vec{A} = (A_x, A_y)$, $\vec{B} = (B_x, B_y)$ two vectors, then

$$\vec{A} \times \vec{B} = (A_x B_y - A_y B_x) \hat{k}$$

✓ The perpendicular unit vectors on the plane of the vectors \vec{A}, \vec{B} is $\frac{\vec{A} \times \vec{B}}{\|\vec{A} \times \vec{B}\|}$

✓ Parallelism of two vectors

The two vectors $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$ are parallel if one of the following conditions occurs:

$$\vec{A} \times \vec{B} = \vec{0} \quad , \quad \frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z} \quad , \quad \vec{A} = k\vec{B}$$

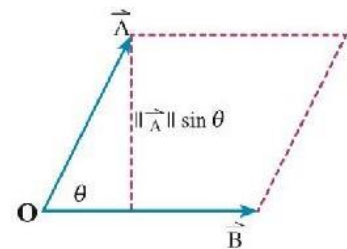
✓ The Geometrical meaning of the vector product of two vectors

$\|\vec{A} \times \vec{B}\| = \text{The area of the parallelogram}$

in which \vec{B} and \vec{A} are two adjacent sides.

$\|\vec{A} \times \vec{B}\| = \text{double the area of triangle}$

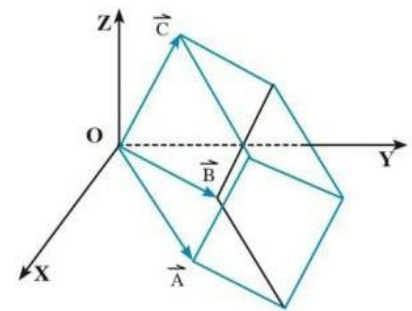
in which \vec{B} and \vec{A} are two adjacent sides.



✓ The scalar triple product $\vec{A} \cdot \vec{B} \times \vec{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

✓ The geometrical meaning of the scalar triple product

The volume of parallelepiped where $\vec{A}, \vec{B}, \vec{C}$ are three vectors representing the non-parallel edges equals the absolute value of $\vec{A} \cdot \vec{B} \times \vec{C}$



📖 Equation of the straight line

📖 Direction vector

i) If ℓ, m, n are the direction cosines of a straight line, then the vectors

$\vec{d} = t(\ell, m, n)$ represents the direction vector of the straight line and is denoted by $\vec{d} = (a, b, c)$ where (a, b, c) are called the direction ratios of the straight line.

ii) The direction vector of the straight line takes different equivalent forms such as

$$\vec{d} = 2(\ell, m, n) = 3(\ell, m, n) = -4(\ell, m, n) \dots\dots\dots$$

Equation of the straight line

- The equation of the straight line which passes through point (x_1, y_1, z_1) and the vector $\vec{d} = (a, b, c)$ is directed vector the vector form: $\vec{r} = (x_1, y_1, z_1) + t(a, b, c)$
- The parametric equations: $x = x_1 + ta, y = y_1 + tb, z = z_1 + tc$
- The Cartesian equation: $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

The angle between two straight lines

If \vec{d}_1 and \vec{d}_2 are the direction vectors of two straight lines, then the smallest angle between the two straight lines is: $\cos \theta = \frac{|\vec{d}_1 \cdot \vec{d}_2|}{\|\vec{d}_1\| \|\vec{d}_2\|}, 0 \leq \theta \leq \frac{\pi}{2}$

and if $(l_1, m_1, n_1), (l_2, m_2, n_2)$ are the direction cosines for the two straight lines, then: $\cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

The parallelism and perpendicularity conditions of two straight lines

If $\vec{d}_1 = (a_1, b_1, c_1)$ and $\vec{d}_2 = (a_2, b_2, c_2)$ are the direction vectors of two straight lines, then

- The two straight lines are parallel if: $\vec{d}_1 = k \vec{d}_2$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ or $\vec{d}_1 \times \vec{d}_2 = \vec{0}$
- The two-straight line are orthogonal if: $\vec{d}_1 \cdot \vec{d}_2 = 0$

The equation of a plane

The equation of the plane passing through point (x_1, y_1, z_1) and the vector $\vec{n} = (a, b, c)$ is perpendicular to the plane .

- ✓ Vector form: $\vec{n} \cdot \vec{r} = \vec{n} \cdot (x_1, y_1, z_1)$
- ✓ Standard form: $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
- ✓ General form: $ax + by + cz + d = 0$

The angle between two planes

If $\vec{n}_1 = (a_1, b_1, c_1), \vec{n}_2 = (a_2, b_2, c_2)$ are the normal vectors to the planes, then the measure of the angle between the two planes is given by the relation :

$$\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|} \text{ where } 0 \leq \theta \leq 90^\circ$$

Parallel and orthogonal planes

If \vec{n}_1 and \vec{n}_2 are the normal vectors to the two planes, then

- i) The two planes are parallel if $\vec{n}_1 // \vec{n}_2 \quad \therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- ii) The planes are perpendicular if $\vec{n}_1 \cdot \vec{n}_2 = 0 \quad \therefore (a_1a_2 + b_1b_2 + c_1c_2) = 0$

The perpendicular length drawn from a point and a plane

Length of the perpendicular drawn from $A(x_1, y_1, z_1)$ to the plane passes through $B(x_2, y_2, z_2)$ and vector $\vec{n} = (a, b, c)$ is perpendicular to the plane.

$$\text{Vector form } L = \frac{|\vec{BA} \cdot \vec{n}|}{\|\vec{n}\|} \quad \text{Cartesian form } L = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Mr. ABD EL HAKIM RASHED

Ministry Evaluation

(Algebra and Solid Geometry)

1-If y-axis cuts the sphere whose center is (3,-4,12) and its radius of length 13 cm at the points A,B then AB=.....

2-Find the area of the triangle whose vertices are A(4,2,-3), B(7,2,-2), C(1,8,-3), then find a unit vector perpendicular to the plane of the triangle.

3-The conjugate of $(i + \omega)$ is

a) $-i + \omega$

b) $-i + \omega^2$

c) $-i - \omega$

d) $-i - \omega^2$

4-If ${}^nC_r = {}^nP_r$, then

a) $r = 0$ or $r \neq 1$

b) $r = 1$ or $r \neq 0$

c) $r = 0$ or $r = 1$

d) $r \neq 0$ or $r = 1$

5-If \vec{A}, \vec{B} are unit vectors and the angle between them is θ

then: $\vec{A} + \vec{B}$ is a unit vector if:

a) $\theta = \frac{\pi}{3}$

b) $\theta = \frac{\pi}{2}$

c) $\theta = \frac{3\pi}{2}$

d) $\theta = \pi$

6-If the $\text{amp.}(Z_1 Z_2) = \frac{\pi}{6}$, $\text{amp.}(Z_1 Z_3) = \frac{2\pi}{9}$

and $\text{amp.}(Z_2 Z_3) = \frac{5\pi}{18}$ then $\text{amp.}(Z_1 Z_2 Z_3) = \dots$

a) $\frac{\pi}{3}$

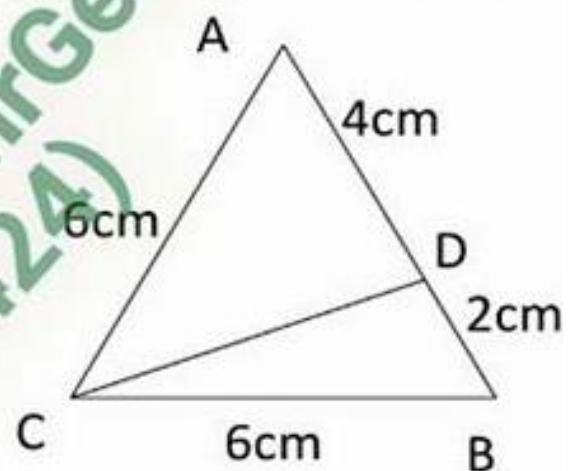
b) $\frac{\pi}{4}$

c) $\frac{\pi}{5}$

d) a) $\frac{\pi}{6}$

7-In the opposite figure Find:

$\overrightarrow{CD} \odot \overrightarrow{CB}$



8-If $Z_1 = 2(\cos 75^\circ + i \sin 75^\circ)$, $Z_2 = 2(\cos 15^\circ + i \sin 15^\circ)$

By using argand digram Find:

i) $Z_1 + Z_2$ in its trig. Form.

ii) $Z_1 - Z_2$ in its exp. form.

9-If $a, b, c \in R$, $a + b + c = 8$, $ab + bc + ac = 12$ then by using properties of deter. Find the value of:

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ 2 & 2 & 2 \end{vmatrix}$$

10-If Z is a complex number and $\text{amp.}(Z - 2) = \frac{\pi}{4}$,

$\text{amp.}(Z - 4) = \frac{2\pi}{4}$, then $\text{amp.}(Z) = \dots$

a) π

b) $\frac{\pi}{2}$

c) $\frac{\pi}{4}$

d) $\frac{2\pi}{4}$

11-If $6 \times {}^nC_3, 4 \times {}^nC_4, 5 \times {}^nC_5$ are three consecutive terms in a G.S then find n .

12-A sphere of center $M(2, -1, -2)$ and radius 3 units is put On a plane $2x + 6y - 3z + k = 0$, then find k .

13-If $k = (2x + i)^{15} - 15(i - 2y)(2x + i)^{14} + \frac{15 \times 14}{2}(i - 2y)^2(2x + i)^{13} + \dots - (i - 2y)^{15}$

Find k if $x = \omega^2 i, y = \omega i$, then find the coefficient of $x^9 y^6$.

14-If $(x - 2) \times {}^nC_3 = {}^nP_3$, then $x = \dots$

a) 5

b) 6

c) 8

d) $3n$

15-In the expansion of $(1 + x)^n$ if :

the coeff. $(T_b) = \text{the coeff. } (T_c)$, then $b + c = \dots$

a) n

b) $n + 2$

c) $n - 2$

d) $2n$

16-In the expansion of $(ax^m + \frac{b}{x^k})^n$, Find the condition which

Make the expansion has a term free of x , $m, k \in \mathbb{R}^+$.

17-If $|Z_1| = |Z_2| = 1$, $\text{amp. } (Z_1 Z_2^5) = 105^\circ$,

$\text{amp. } (Z_1 \div Z_2) = 33^\circ$ and $x + yi = Z_1^{14} + Z_2^{15}$

then find x, y .

18-Without expansion Prove that:

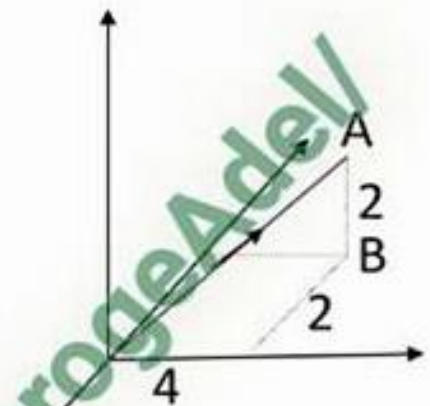
$$\begin{vmatrix} a & b & 1 \\ a^2 & b^2 & a+b \\ a^3 & b^3 & a^2+ab+b^2 \end{vmatrix} = 0$$

19-In the figure opposite:

If $f = 12\sqrt{26}$ new

i) Find the components of \vec{F}

ii) Find the work done by the force to move a body from O to B.



20-A sphere is put inside a cube the length of its edge is 10 cm

And touch its faces if one of its verticiec is the origin then

The equation of the sphere is.....

a) $x^2 + y^2 + z^2 - 20x - 20y - 20z + 50 = 0$

b) $x^2 + y^2 + z^2 + 10x + 10y + 10z + 50 = 0$

c) $x^2 + y^2 + z^2 - 10x - 10y - 10z + 50 = 0$

d) $x^2 + y^2 + z^2 - 20xy - 20vy - 20xz + 50 = 0$

21-The equation of the plane which cuts x,y,z axes resp. by

2,-3,4 is.....

a) $2x - 3y + 4z = 1$

b) a) $6x - 4y + 3z = 6$

c) $6x + 4y + 3z = 12$

d) $6x - 4y + 3z = 12$

22- ${}^{n+1}C_3 + {}^{n+1}C_2 + {}^{n+1}C_1 = \dots$

a) ${}^{n+1}C_3$

b) ${}^{n+1}C_2$

c) ${}^{n+1}C_1$

d) ${}^{n+3}C_3$

23-The number of ways to form a number less than 300 from the set $\{1, 2, 5, 6\}$

24- $\frac{3}{2+\omega} = \dots$

a) $2 - \omega$

b) $2 + \omega^2$

c) $3(1 + \omega)^3$

d) $3\omega^2$

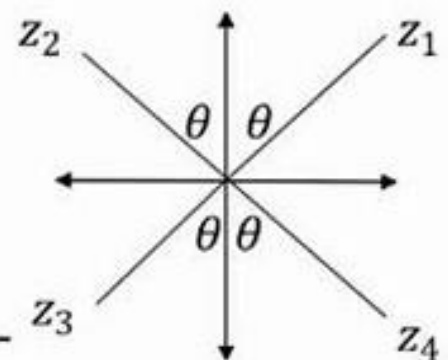
25-If Z is a complex number then find the S.S of:

$$2z - 3\bar{z} = 5 + 10i$$

26- If z_1, z_2, z_3, z_4 are complex

numbers then

$$\text{amp.}(z_1 z_2 z_3 z_4) = \dots\dots\dots$$



27-If $A^{-1} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ then $y = \dots\dots\dots$

29-Prove that:

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc + bc + ca + ab$$

30-Use matrices to solve the equations:

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

31- If $\vec{A} = \vec{B} \times \vec{C}$, $\vec{A}, \vec{B}, \vec{C}$ non zeroes vectors then:

$$\vec{A} \odot \vec{B} = \dots$$

32-Find the equation of the straight line passing through

(2,-1,3) and intersect the straight line

$\vec{r} = (1, -1, 2) + k(3, 2, -1)$ orthogonally.

GOOD LUCK

① Model answer
of (Ministry Eval.)
(alg., Solid)

$$\text{II] } (x-3)^2 + (y+4)^2 + (z-12)^2 = (13)^2$$

Points of inter. with
y-axis
(0, y, 0)

$$\therefore (0-3)^2 + (y+4)^2 + (0-12)^2 = (13)^2$$

$$\therefore 9 + (y+4)^2 + 144 = 169$$

$$(y+4)^2 = 16$$

$$y+4 = \pm 4$$

$$\begin{cases} y+4=4 \\ y+4=-4 \end{cases} \Rightarrow \begin{cases} y=0 \\ y=-8 \end{cases}$$

\therefore The Points of inter.
are (0, 0, 0), (0, -8, 0)

(A)

(B)



$$AB = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$= \sqrt{0 + 64 + 0} = \sqrt{64} = 8 \text{ unit length}$$

2) $(4, 2, -3)$ A
 $\vec{AB} = \vec{B} - \vec{A} = (3, 4, 1)$ B (7, -2, -2) C (1, 8, -3)

$$\vec{AC} = \vec{C} - \vec{A} = (-3, 6, 0)$$

$$\|\vec{AB} \times \vec{AC}\| = 2 \text{ area of } \Delta ABC$$

$$\begin{aligned} \therefore \vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 1 \\ -3 & 6 & 0 \end{vmatrix} \\ &= (-6)\hat{i} - (3)\hat{j} + (18)\hat{k} \\ &= (-6, -3, 6) \end{aligned}$$

$$\|\vec{AB} \times \vec{AC}\| = \sqrt{36 + 9 + 36} = 9$$

$$\therefore 9 = 2 \text{ area of } \Delta ABC$$

$$\therefore \text{area of } \Delta ABC = \boxed{4.5} \text{ sq unit}$$

$$\therefore \vec{n} = \vec{AB} \times \vec{AC} = (-6, -3, 6)$$

$$\begin{aligned} \therefore \vec{n}^* &= \frac{\vec{n}}{\|\vec{n}\|} \\ &= \frac{(-6, -3, 6)}{9} \\ &= \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) \end{aligned}$$

(2) 3 b

because

$$\text{Sum} = i + \omega - i + \omega^2 \\ = \omega + \omega^2 = \boxed{-1} \in \mathbb{R}$$

$$\text{Prod.} = (i + \omega)(-i + \omega^2) \\ = -i^2 + i\omega^2 - i\omega + \omega^3 \\ = 1 + i(\omega^2 - \omega) + 1 \\ = 2 + i(\pm\sqrt{3}i) \\ = 2 + (\pm\sqrt{3})(-1)$$

$$\therefore (i + \omega), (-i + \omega^2) \in \mathbb{R} \\ \text{Conj.}$$

4 c

$$nC_r = nP_r$$

$$\frac{nP_r}{\sqrt{r}} = nP_r$$

$$\sqrt{r} = 1 \Rightarrow \begin{cases} r=0 \\ r=1 \end{cases} \in \mathbb{R}$$

5 $\|\vec{A}\| = \|\vec{B}\| = 1$

$\therefore \vec{A} + \vec{B}$ is a unit vect.

$$\therefore \|\vec{A} + \vec{B}\| = 1 \text{ (sq.)}$$

$$\|\vec{A} + \vec{B}\|^2 = 1$$

$$(\vec{A} + \vec{B}) \odot (\vec{A} + \vec{B}) = 1$$

$$\vec{A} \odot \vec{A} + \vec{A} \odot \vec{B} + \vec{B} \odot \vec{A} + \vec{B} \odot \vec{B} = 1$$

$$\|\vec{A}\|^2 + 2(\vec{A} \odot \vec{B}) + \|\vec{B}\|^2 = 1$$

$$1 + 2(\vec{A} \odot \vec{B}) + 1 = 1$$

$$\therefore (\vec{A} \odot \vec{B}) = -\frac{1}{2}$$

$$\therefore \|\vec{A}\| \|\vec{B}\| \cos \theta = -\frac{1}{2}$$

$$(1)(1) \cos \theta = -\frac{1}{2}$$

$$\therefore m(\angle \theta) = 120^\circ \\ = \boxed{\frac{2\pi}{3}}$$

6 Let $Z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$
 $Z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$
 $Z_3 = r_3 (\cos \theta_3 + i \sin \theta_3)$

$$\therefore \text{amp.}(Z_1 Z_2) = \frac{\pi}{6}$$

$$\therefore (\theta_1 + \theta_2 = 30^\circ) \rightarrow (1)$$

$$\therefore \text{amp.}(Z_1 Z_3) = \frac{2\pi}{9}$$

$$\therefore (\theta_1 + \theta_3 = 40^\circ) \rightarrow (2)$$

$$\therefore \text{amp.}(Z_2 Z_3) = \frac{5\pi}{18}$$

$$\therefore (\theta_2 + \theta_3 = 50^\circ) \rightarrow (3)$$

by adding (1), (2), (3)

$$\therefore 2(\theta_1 + \theta_2 + \theta_3) = 120^\circ$$

$$\therefore \theta_1 + \theta_2 + \theta_3 = \frac{60^\circ}{1} \\ = \boxed{\frac{\pi}{3}}$$

③

7

In $\triangle DBN$

$$\cos 60^\circ = \frac{BN}{2}$$

$$BN = 1$$

$$\sin 60^\circ = \frac{DN}{2}$$

$$DN = \sqrt{3}$$

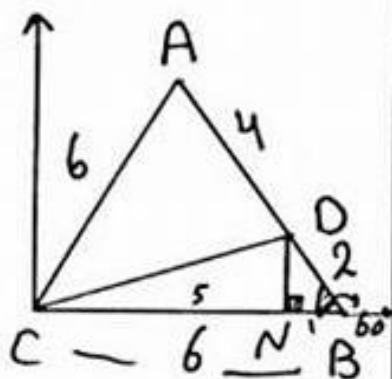
$$\therefore C(0,0), B(6,0)$$

$$D(5, \sqrt{3})$$

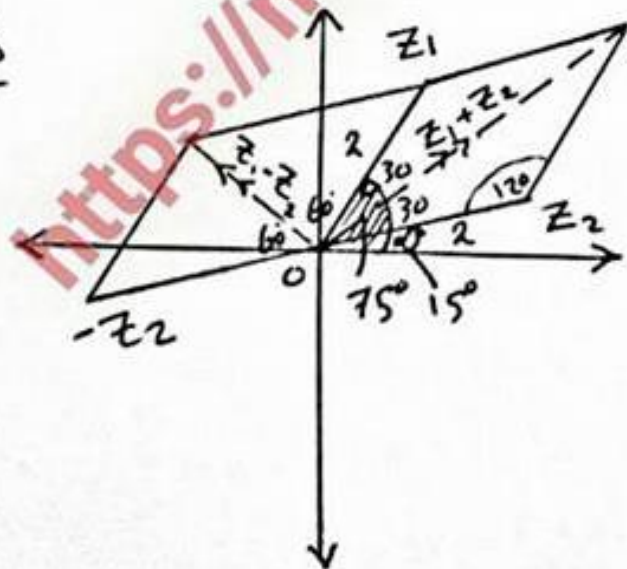
$$\therefore \vec{CD} = \vec{D} - \vec{C} = (5, \sqrt{3})$$

$$\vec{CB} = \vec{B} - \vec{C} = (6,0) - (0,0) = (6,0)$$

$$\therefore \vec{CD} \cdot \vec{CB} = (5, \sqrt{3}) \cdot (6,0) = 30$$



8



$z_1 + z_2$ represented by the diagonal of the rhombus.

whose length = $2\sqrt{3}$

and its amp. = $15^\circ + 30^\circ = 45^\circ$



$$\therefore z_1 + z_2 = 2\sqrt{3} (\cos 45^\circ + i \sin 45^\circ)$$

Similarly

$$z_1 - z_2 = z_1 + (-z_2)$$

which repr. by the diag. of the second rhombus

$$\therefore z_1 - z_2 = 2 (\cos 135^\circ + i \sin 135^\circ)$$

9

$$\therefore a + b + c = 8 \text{ (by sq)}$$

$$(a+b+c)^2 = 64$$

$$[(a+b)+c]^2 = 64$$

$$(a+b)^2 + 2(a+b)c + c^2 = 64$$

$$a^2 + 2ab + b^2 + 2ac + 2bc + c^2 = 64$$

$$\therefore a^2 + b^2 + c^2 + 2(ab+ac+bc) = 64$$

④ ↓

$$\therefore a^2 + b^2 + c^2 + 2(12) = 64$$

$$a^2 + b^2 + c^2 + 24 = 64$$

$$a^2 + b^2 + c^2 = \textcircled{40}$$

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ 2 & 2 & 2 \end{vmatrix}$$

$$= 2 \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ 1 & 1 & 1 \end{vmatrix}$$

$$(R_1 \times -1) + R_2$$

$$= 2 \begin{vmatrix} a & b & c \\ -b & -c & -a \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} a & b & c \\ b & c & a \\ 1 & 1 & 1 \end{vmatrix}$$

$$(C_1 \times -1) + C_2, (C_1 \times -1) + C_3$$

$$= -2 \begin{vmatrix} a & b-a & c-a \\ b & c-b & a-b \\ 1 & 0 & 0 \end{vmatrix}$$

$$= -2 \begin{vmatrix} b-a & c-a \\ c-b & a-b \end{vmatrix}$$

$$= -2 [(b-a)(a-b) - (c-a)(c-b)]$$

$$= -2 [ab - b^2 - a^2 + ab - (c^2 - cb - ac + ab)]$$

$$= -2 [ab - b^2 - a^2 + ab - c^2 + cb + ac - ab]$$

$$= 2 [a^2 + b^2 + c^2 - (ab + cb + ac)]$$

$$= 2 [40 - 12]$$

$$= 2 [28] = \textcircled{56}$$

10 Let $z = x + yi$

$$\therefore z - 2 = x + yi - 2$$

$$= (x-2) + yi$$

$$\therefore \tan \theta = \frac{y}{x-2}$$

$$\therefore \tan \left(\frac{\pi}{2} \right) = \frac{y}{x-2}$$

$$\frac{1}{0} = \frac{y}{x-2}$$

$$\therefore \textcircled{x=2} \rightarrow \textcircled{1}$$

$$z - 4 = x + yi - 4$$

$$= (x-4) + yi$$

$$\therefore \tan \theta = \frac{y}{x-4}$$

$$\tan(135^\circ) = \frac{y}{x-4}$$

$$\downarrow -1 = \frac{y}{-2} \Rightarrow \textcircled{y=2}$$

⑤ $z = 2 + 2i$
 $\therefore \tan \theta = 1 \Rightarrow \theta = \left(\frac{\pi}{4}\right)$

⑪ $\therefore G.S$

$\therefore (4x^n C_4)^2 = (6x^n C_3) \times (5x^n C_5)$

$\therefore 16({}^n C_4)({}^n C_4) = 30({}^n C_3)({}^n C_5)$

$\therefore \frac{{}^n C_5}{{}^n C_4} = \left(\frac{16}{30}\right) \left(\frac{{}^n C_4}{{}^n C_3}\right)$

$\frac{n-5+1}{5} = \frac{8}{15} \left(\frac{n-4+1}{4}\right)$

$\frac{n-4}{5} = \frac{28}{15} \left(\frac{n-3}{4}\right)$

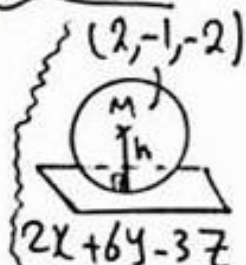
$3(n-4) = 2(n-3)$

$3n - 12 = 2n - 6$

$n = 6$

⑫

$h = r = 3$



$\therefore \frac{|2(2) + 6(-1) - 3(2) + k|}{\sqrt{4 + 36 + 9}} = 3$

$3 = \frac{|4 - 6 + 6 + k|}{7}$

$21 = |k + 4|$

$k + 4 = \pm 21$

$k + 4 = 21 \quad | \quad k + 4 = -21$
 $k = 17 \quad | \quad k = -25$

⑬

$k = [(2x + i) - (i - 2y)]^{15}$
 $= [2x + i - i + 2y]^{15}$
 $= (2x + 2y)^{15}$

$\therefore x = \omega^2 i, y = \omega i$

$\therefore k = (2)^{15} (x + y)^{15}$
 $= (2)^{15} (\omega^2 i + \omega i)^{15}$
 $= (2)^{15} (i(\omega^2 + \omega))^{15}$
 $= (2)^{15} (i)^{15} (-1)^{15}$
 $= -(2)^{15} (i^3)$
 $= (2)^{15} (i)$

T_7 has $x^9 y^6$

$\therefore T_7 = {}^{15}C_6 (2y)^6 (2x)^9$
 $= {}^{15}C_6 (2)^{15} (y)^6 (x)^9$

$\therefore \text{Coeff. of } T_7$
 $= {}^{15}C_6 (2)^{15}$
 $= \boxed{=}$

(6) 14 $(x-2) \times \frac{nP_3}{13} = nP_3$
 $\therefore (x-2) = 13$
 $(x-2) = 6$
 $\boxed{x=8}$

15 $T_b = {}^nC_{b-1} (x)^{b-1}$
 $\therefore \text{Coeff. } (T_b) = {}^nC_{b-1}$
 and $\text{Coeff. } (T_c) = {}^nC_{c-1}$
 $\therefore {}^nC_{b-1} = {}^nC_{c-1}$
 $\therefore b-1 + c-1 = n$
 $\therefore b+c = \boxed{n+2}$

16 $T_{r+1} = {}^nC_r \left(\frac{b}{x^k}\right)^r (ax^m)^{n-r}$
 $= {}^nC_r (b)^r (x)^{-kr} (a)^{n-r} (x)^{nm-mr}$
 $= {}^nC_r (b)^r (a)^{n-r} (x)^{-kr+nm-mr}$
 $\therefore -kr+nm-mr=0$
 $\therefore r(k+m) = nm$
 $r = \frac{nm}{k+m}$

\therefore The condition is
 nm is divisible by
 $(k+m)$

17 let $Z_1 = \cos \theta_1 + i \sin \theta_1$
 $Z_2 = \cos \theta_2 + i \sin \theta_2$
 $\text{amp. } (Z_1 Z_2) = 105^\circ$
 $\therefore \boxed{\theta_1 + \theta_2 = 105^\circ}$
 $\text{amp. } (Z_1 \div Z_2) \rightarrow \text{C}$
 $= 33^\circ$
 $\therefore \boxed{\theta_1 - \theta_2 = 33^\circ} \rightarrow \text{D}$

by Sub.

$6\theta_2 = 72$
 $\therefore \theta_2 = \boxed{12^\circ}$

$\therefore \boxed{\theta_1 = 45^\circ}$

$\therefore x+yi = Z_1^{14} + Z_2^{15}$
 $= \cos(14 \times 45) + i \sin(14 \times 45) + \cos(12 \times 15) + i \sin(12 \times 15)$
 $= \cos 270 + i \sin 270 + \cos 180 + i \sin 180$
 $= 0 - i - 1 + 0$
 $\therefore \boxed{x=-1}, \boxed{y=-1}$

7

$$\begin{aligned} \text{L.H.S} &= \begin{vmatrix} a & b & 1 \\ a^2 & b^2 & a+b \\ a^3 & b^3 & a^2+ab+b^2 \end{vmatrix} \\ &= (C_2 \times -1) + C_1 \\ &= \begin{vmatrix} a-b & b & 1 \\ a^2-b^2 & b^2 & a+b \\ a^3-b^3 & b^3 & a^2+ab+b^2 \end{vmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{vmatrix} a-b & b & 1 \\ (a-b)(a+b) & b^2 & a+b \\ (a-b)(a^2+ab+b^2) & b^3 & a^2+ab+b^2 \end{vmatrix} \\ &= (a-b) \begin{vmatrix} 1 & b & 1 \\ a+b & b^2 & a+b \\ a^2+ab+b^2 & b^3 & a^2+ab+b^2 \end{vmatrix} \\ &C_1 = C_3 \end{aligned}$$

$$\begin{aligned} \therefore \text{L.H.S} &= (a-b)(\text{Zero}) \\ &= \text{Zero} = \text{R.H.S} \end{aligned}$$

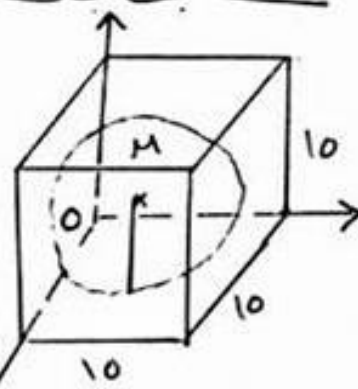
$$\begin{aligned} \text{19} \quad A &= (-2, 4, 3) \quad (\text{Corr.}) \\ B &= (-2, 4, 0) \\ \vec{F} &= \frac{\|\vec{F}\|}{\|\vec{F}\|} \vec{OA}^* \\ &= \frac{(12\sqrt{29})}{\sqrt{4+16+9}} \times \frac{(-2, 4, 3)}{\sqrt{29}} \\ &= (12\sqrt{29}) \times \frac{(-2, 4, 3)}{\sqrt{29}} \end{aligned}$$

$$= (-24, 48, 36)$$

$$\vec{OB} = \vec{B} - \vec{O} = (-2, 4, 0)$$

$$\begin{aligned} W &= \vec{F} \cdot \vec{OB} \\ &= (-24, 48, 36) \cdot (-2, 4, 0) \\ &= 48 + 192 \\ &= \boxed{240} \text{ unit of work} \end{aligned}$$

$$\begin{aligned} \text{20} \quad C \\ M &= (5, 5, 5) \\ V &= 5 \end{aligned}$$



$$\therefore x^2 + y^2 + z^2 - 10x - 10y - 10z + 50 = 0$$

$$\text{21} \quad \frac{x}{2} + \frac{y}{-3} + \frac{z}{4} = 1 \quad (\times 12)$$

$$6x - 4y + 3z = 12$$

Good Luck


Mr/ George Adel

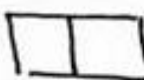
⑧ 22 d)

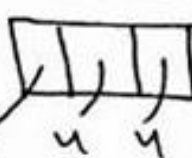
$${}^{n+1}C_3 + {}^{n+1}C_2 + {}^{n+1}C_1$$

$$= {}^{n+1}C_3 + {}^{n+2}C_2$$

$$= {}^{n+3}C_3$$

23 4 P₁ = (4) 

4 x 4 = (16) 

2 x 4 x 4 = (32) 

The number of ways

$$= 4 + 16 + 32 = (52)$$

24 $\frac{3}{2+w} = \frac{b}{2+w^2}$

$$\frac{3}{2+w} \times \frac{2+w^2}{2+w^2}$$

$$= \frac{3(2+w^2)}{4+2w^2+2w+w^3}$$

$$= \frac{3(2+w^2)}{4+2(w^2+w)+1}$$

$$= \frac{3(2+w^2)}{5+2(-1)}$$

$$= \frac{3(2+w^2)}{3}$$

$$= \boxed{2+w^2}$$

25 Let $z = a + bi$

$$2z - 3\bar{z} = 5 + 10i$$

$$2(a+bi) - 3(a-bi) = 5 + 10i$$

$$2a + 2bi - 3a + 3bi = 5 + 10i$$

$$-a + 5bi = 5 + 10i$$

$$\begin{cases} -a = 5 \\ a = -5 \end{cases} \quad \begin{cases} 5b = 10 \\ b = 2 \end{cases}$$

$$z = -5 + 2i$$

26 $\arg(z_1) = 90^\circ - \theta$

$$\arg(z_2) = 90^\circ + \theta$$

$$\arg(z_3) = 270^\circ - \theta$$

$$\arg(z_4) = 270^\circ + \theta$$

Then

$$\arg(z_1 z_2 z_3 z_4)$$

$$= 90^\circ - \theta + 90^\circ + \theta + 270^\circ - \theta + 270^\circ + \theta = \boxed{720^\circ}$$

27

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+10 \\ 1+6 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

$$\therefore y = 7$$

9

29

$$L.H.S = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} + \begin{vmatrix} a & 1 & 1 \\ 0 & 1+b & 1 \\ 0 & 1 & 1+c \end{vmatrix}$$

$$(R_1 \times -1) + R_2, (R_1 \times -1) + R_3$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$+ a [(1+b)(1+c)]$$

$$= bc + a [1+c+b+bc]$$

$$= bc + ac + ab + abc$$

$$= R.H.S$$

30 let $\frac{1}{x} = L, \frac{1}{y} = M$

$$\frac{1}{z} = N$$

$$\therefore \begin{cases} 2L + 3M + 10N = 4 \\ 4L - 6M + 5N = 1 \\ 6L + 9M - 20N = 2 \end{cases}$$

$$\begin{pmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{pmatrix} \begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

↓ complete

$$\begin{pmatrix} L \\ M \\ N \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{pmatrix}$$

$$\therefore (x=2), (y=3), (z=5)$$

31 $\vec{B} \times \vec{C}$ is a vector
⊥ Plane of them

$$\therefore \vec{A} = (\vec{B} \times \vec{C})$$

$$\therefore \vec{A} \perp \text{Plane of } \vec{B}, \vec{C}$$

$$\therefore \vec{A} \cdot \vec{B} = \text{Zero}$$

32

$$\text{Let } C \in L_2$$

$$C = (1+3k, -1+2k, 2-k) \quad \begin{cases} \vec{r} = (1, -1, 2) \\ + k(3, 2, -1) \end{cases}$$

$$\vec{AC} = \vec{C} - \vec{A}$$

$$= (-1+3k, 2k, -1-k)$$

$$\vec{AC} \cdot \vec{d} = 0$$

$$(-1+3k, 2k, -1-k) \cdot (3, 3, -1) = 0$$

$$-3+9k+6k+1+k=0$$

$$14k=2 \rightarrow k = \left(\frac{1}{7}\right)$$

$$\therefore \vec{AC} = \left(-\frac{4}{7}, \frac{2}{7}, -\frac{8}{7}\right)$$

$$\therefore \vec{r} = (2, -1, 3) + t\left(-\frac{4}{7}, \frac{2}{7}, -\frac{8}{7}\right)$$

$$\vec{r} = (2, -1, 3) + t'(2, -1, 4)$$

