Rule Algebra
3rd Sec .
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Mr.Fat7y
01002627894

Chapter 1

1] The three different ordered pairs formed by 1 , 2 , 3

Are (1 , 2) , (1 , 3) , (2 , 3) , (2 , 1) , (3 , 1) , (3 , 2)

${}^{3}P_{2} = 3 \times 2 = 6$

2] The three sets consist of two elements formed by 1 , 2 , 3

Are { 1 , 2 } , { 1 , 3 } , { 2 , 3 }

${}^{3}C_{2} = 3$

Rule of permutations .

3] $^nP_r = n(n-1)(n-2) \ldots\ldots\ldots (n-r+1)$ such that $n, r \in Z^+$

where $1 \le r \le n$ use this rule when r is known and small .

4] $\lfloor n = {}^nP_n = n(n-1)(n-2) \ldots \times 3 \times 2 \times 1$ factorial n

5] $\lfloor 10 = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 10 \times \lfloor 9$

$= 10 \times 9 \times \lfloor 8$

$= 10 \times 9 \times 8 \times \lfloor 7$

$= 10 \times 9 \times 8 \times 7 \times \lfloor 6$

$= 10 \times 9 \times 8 \times 7 \times 6 \times \lfloor 5$

6] $\lfloor n = n \lfloor n-1 = n(n-1) \lfloor n-2$

7] $^nP_r = \dfrac{\lfloor n}{\lfloor n-r}$ 8] $^nP_0 = \lfloor 0 = 1$ 9] $\lfloor 1 = 1$

Rule of combinations

1] $^nC_r = \dfrac{^nP_r}{\lfloor r} = \dfrac{\lfloor n}{\lfloor n-r \lfloor r}$ such that $n, r \in Z^+$ and $r \le n$

2] $^nC_0 = {}^nC_n = 1$ 3] $^nC_1 = n$

4] $^nC_r = {}^nC_{n-r}$ using this rule when r > half n

Ex If $^nC_r = {}^nC_5$ $\therefore r = 5$ or $r + 5 = n$

5] $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

6] $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

The binomial theorem

1] $(x + a)^1 = x + a$

2] $(x + a)^2 = x^2 + 2ax + a^2$

3] $(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$

4] $(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$

5] $(x + a)^5 = x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5$

$= {}^5C_0\, x^5 + {}^5C_1\, a\, x^4 + {}^5C_2\, a^2 x^3 + {}^5C_3\, a^3 x^2 + {}^5C_4\, a^4 x + {}^5C_5\, a^5$

6] $(1 + x)^n = 1 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots\ldots + x^n$ according to ascending power of x

7] $(1 - x)^n = 1 - {}^nC_1\, x + {}^nC_2\, x^2 + \ldots\ldots + (-x)^n$ according to ascending power of x

8] $(x + 1)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2} + \ldots\ldots + {}^nC_n$

according to descending power of x

9] $(x + a)^n = {}^nC_0\, x^n + {}^nC_1\, a\, x^{n-1} + {}^nC_2\, a^2 x^{n-2} + \ldots\ldots + {}^nC_n\, a^n$

$\boxed{T_{r+1} = {}^nC_r\, a^r x^{n-r} = {}^nC_r\,(\text{second})^r\,(\text{first})^{n-r}}$ where $r = 0, 1, 2, \ldots\ldots, n$

10] At $(x + a)^n$ then power of x is descending but in $(a + x)^n$ the power of x is ascending .

11] The sum of the terms coefficients of the expansion of $(2x + 3y)^5 = (2 + 3)^5 = 3125$

We get this result by putting $x = 1$ & $y = 1$.

Remarks

12] number of terms $n + 1$

13] power of x descending but power of a ascending

14] If n is even , then there exist one middle term whose order is $\frac{n}{2} + 1$

15] If n is odd then there exist two middle terms whose orders are $\frac{n+1}{2}$ and $\frac{n+3}{2}$

16] $(x + a)^n + (x - a)^n = 2\,[T_1 + T_3 + T_5 + \ldots\ldots]$

17] $(x + a)^n - (x - a)^n = 2\,[T_2 + T_4 + T_6 + \ldots\ldots]$

18] In the expansion of $(x + a)^n$, we have $\frac{T_{r+1}}{T_r} = \frac{n - r + 1}{r} \times \frac{a}{x}$

Chapter 2

Complex numbers C

1] $i = \sqrt{-1}$

$i = i$	$i^2 = -1$	$i^3 = -i$	$i^4 = 1$
$i^5 = i$	$i^6 = -1$	$i^7 = -i$	$i^8 = 1$
$i^9 = i$	$i^{10} = -1$	$i^{11} = -i$	$i^{12} = 1$

2] $C = \{\, x + i\,y ,\ x , y \in R ,\ i^2 = -1 \,\}$

3] $z = x + i\,y$ complex number
x is the real part , y is the imaginary part .

4] Two complex numbers $x_1 + y_1 i$, $x_2 + y_2 i$ are equal iff $x_1 = x_2$ and $y_1 = y_2$

5] Their sum $= (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2) i$

6] Their product $= (x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i$

7] If $x + y i = 0$ then $x = 0$, $y = 0$

8] The additive inverse of the number $z = x + y i$ is $-z = -x - y i$

$\therefore Z + (-Z) = 0$

9] $z = x + y i$ its conjugate $\overline{z} = x - y i$

10] If $z = x + y i$ is one of the roots of the equation $ax^2 + bx + c = 0$

Where a , b , c $\in$ R , then the other root is $\overline{z} = x - y i$.

11] $z + \overline{z} = (x + iy) + (x - iy) = 2x$

12] $z\overline{z} = (x + iy)(x - iy) = x^2 + y^2$

13) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

14] $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$

15] $\overline{\overline{z}} = z$

16] $\overline{z_1 \div z_2} = \overline{z_1} \div \overline{z_2}$

17] $x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$ to solve $ax^2 + bx + c = 0$

18] Find the S.S. for the equation $x^2 - 6x + 13 = 0$ $\therefore a = 1$, $b = -6$, $c = 13$

$x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = 3 \pm 2i$ $\therefore$ S.S. = { 3 + 2 i , 3 − 2 i }

19] $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ for any $z_1 , z_2 , z_3 \in$ R

20] $1 \times z = z \times 1 = z$

1 is the identity element with respect to multiplication .

21] $z + 0 = 0 + z = z$

zero is the identity element with respect to addition

22] $z_1 z_2 = z_2 z_1$ Commutative law of multiplication

23] $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ Associative law of multiplication

24] The multiplicative inverse of $z = x + iy$

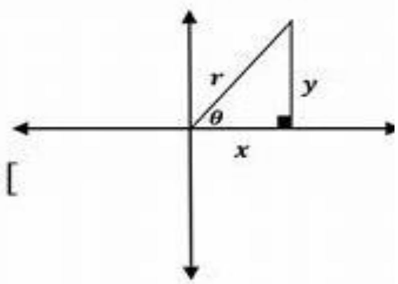
$$\frac{1}{z} = \frac{1}{x + iy} \times \frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$$

25] $L^2 + M^2 = (L + M)^2 - 2\,LM$

26] $(L - M)^2 = (L + M)^2 - 4\,LM$

Argand diagram

The Cartesian coordinates of the point $A(x , y)$ can be converted into the polar form $A(r , \theta)$ where $r = |z| = \sqrt{x^2 + y^2}$ & $\tan\theta = \frac{y}{x}$

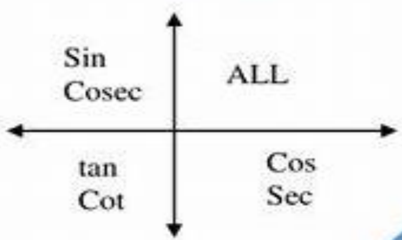


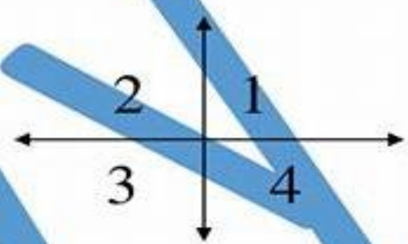
represent a complex number $z = x + y\,i$ (algebraic form)

$z = r\,(\cos\theta + i\,\sin\theta)$ (**trig. form**) where $\theta \in [-\pi , \pi[$

θ **is called amplitude of the complex number.**

r **is called modulus of the complex number.**





Euler form

1] $\sin x = \frac{x}{\underline{|1}} - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \ldots$

2] $\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \ldots$

3] $e^x = 1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots$

4] $e^x = \cos x + i\,\sin x$

5] $z = x + y\,i = r\,(\cos\theta + i\,\sin\theta) = r\,e^{\theta i}$

Remarks :

1a] $1 = \cos 0 + i\,\sin 0 = e^{0\,i}$

b] $i = \cos 90 + i\,\sin 90 = e^{\frac{\pi}{2}i}$

c] $-1 = \cos\pi + i\,\sin\pi = e^{\pi i}$

d] $-i = \cos 270 + i\,\sin 270 = e^{\frac{-\pi}{2}i}$

2] $|z_1 z_2| = |z_1|\,|z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

3] If $Z = r\,(\cos\theta + i\,\sin\theta)$ then $Z^2 = r^2\,(\cos 2\theta + i\,\sin 2\theta) = r^2 e^{2\theta i}$

And also $Z^n = r^n\,(\cos n\theta + i\,\sin n\theta) = r^n e^{n\theta i}$

4] If $Z = r\,(\cos\theta + i\,\sin\theta)$ $\therefore \frac{1}{z} = \frac{1}{r}\,[\cos(-\theta) + i\,\sin(-\theta)] = r^{-1} e^{-\theta i}$

5] If $z_1 = r_1\,(\cos\theta_1 + i\,\sin\theta_1)$ and $z_2 = r_2\,(\cos\theta_2 + i\,\sin\theta_2)$

then $z_1 z_2 = r_1 r_2 [\, Cos\,(\theta_1 + \theta_2) + i\, Sin\,(\theta_1 + \theta_2)\,] = r_1 r_2 e^{(\theta_1 + \theta_2)\, i}$

and $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\, Cos\,(\theta_1 - \theta_2) + i\, Sin\,(\theta_1 - \theta_2)] = \frac{r_1}{r_2} e^{(\theta_1 - \theta_2)\, i}$

6] $\frac{z_1 z_2}{z_3} = \frac{r_1 r_2}{r_3} \,[\, Cos\,(\theta_1 + \theta_2 - \theta_3) + i\, Sin\,(\theta_1 + \theta_2 - \theta_3)\,] = \frac{r_1 r_2}{r_3}\, e^{(\theta_1 + \theta_2 - \theta_3)\, i}$

7] $e^{i\theta}$ and $e^{-i\theta}$ are conjugate

8] $r_1 e^{\theta_1 i} \times r_2 e^{\theta_2 i} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

9] $\frac{r_1 e^{\theta_1 i}}{r_2 e^{\theta_2 i}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

10] $(re^{\theta i})^n = r^n e^{n\theta i}$

11] $\sqrt[n]{re^{\theta i}} = \sqrt[n]{r} e^{\frac{\theta + 2\pi m}{n} i}$ Where $m = 0, 1, 2, \ , n-1$

12] $\overline{re^{\theta i}} = \bar{r} e^{\frac{\theta + 2\pi r}{n} i}$ where $r \in \{0, 1, 2, 3, \ldots\ldots\ldots, (n-1)\}$

De moiver's theorem

If n is a rational number, $Z = r\,(Cos\,\theta + i\,Sin\,\theta)$:

$\therefore Z^n = r^n (Cos\,\theta + i\,Sin\,\theta)^n = r^n (Cos\,n\theta + i\,Sin\,n\theta)$ if $n = \frac{1}{k}$

$\sqrt[k]{z} = z^{\frac{1}{k}} = r^{\frac{1}{k}} (Cos\frac{\theta + 2m\pi}{k} + i\,sin\frac{\theta + 2m\pi}{k}) = r^{\frac{1}{k}} e^{\frac{\theta + 2m\pi}{k}}$ Where $m = 0, \mp 1, \mp 2, \ldots\ldots$

Remarks

If $\sqrt{a + bi} = x + y\,i$ by squaring both sides .

$\therefore a + bi = x^2 - y^2 + 2xyi \quad \therefore a = x^2 - y^2 \;,\; b = 2xy$

The cubic root of 1 :

1] When we solve $x^3 = 1$ $\quad \therefore x^3 - 1 = (x - 1)(x^2 - x + 1)$

then $x = 1\,,\, x = \frac{-1}{2} + \frac{\sqrt{3}}{2} i \,,\, x = \frac{-1}{2} - \frac{\sqrt{3}}{2} i$

2] The cubic roots of one are $1, w, w^2$.

3] Their angles are 0^0 , 120^0 , 240^0 .

4] the squaring of any one of the two complex root equal the other .

5] $w - w^2 = \pm\sqrt{3}\,i$ (you must prove in each example)

6] $1 + w + w^2 = 0$

7] $w^3 = 1$

8] $\frac{1}{W} = W^2$

9] $\frac{1}{W^2} = W$

10] $W - W^2 = \pm\sqrt{3}\,i$

11] $W^2 - W = \mp\sqrt{3}\,i$

Chapter 3

Determinants

1] The determinant consists of some elements that are arranged in ((n)) rows and ((n)) columns

For example $\begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix}$ is a second order determinant .

Also $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & -1 \\ -2 & -2 & 5 \end{vmatrix}$ is a third order determinant .

2] $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

Example $\begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} = 2 \times 8 - 5 \times 3 = 1$

Example $\begin{vmatrix} 1 & w \\ w^2 & -w \end{vmatrix} = 1 \times -w - w \times w^2 = -w - w^3 = -w - 1 = -(w+1) = -(-w^2) = w^2$

3] The sign of any element cofactor in 3rd degree determinant can be found from the following figure .

$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Some properties of determinants

4] If the rows are replaced by the columns and the columns by the rows in the same order , then the value of the determinant is unchanged

$\begin{vmatrix} 1 & -3 & 5 \\ 4 & 0 & 7 \\ 8 & 14 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 8 \\ -3 & 0 & 14 \\ 5 & 7 & -3 \end{vmatrix}$

5] We can find the value of the determinant using any row or any column

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6] If all the elements of any row (column) in determinant are zeroes , then the value of the determinant is zero .

$\begin{vmatrix} 1 & 2 & 9 \\ 0 & 0 & 0 \\ 3 & 8 & 2 \end{vmatrix} = 0$, $\begin{vmatrix} 0 & 2 & 9 \\ 0 & 7 & 4 \\ 0 & 8 & 2 \end{vmatrix} = 0$

7] If the corresponding elements in two rows (columns) are equal , then the value of the determinant is zero .

$$\begin{vmatrix}1&4&7\\6&5&8\\6&5&8\end{vmatrix}=0 \quad , \quad \begin{vmatrix}1&4&4\\2&5&5\\3&6&6\end{vmatrix}=0$$

8] If two rows or two columns are interchanged , then the resulting determinant = − original determinant .

$$\begin{vmatrix}1&4&7\\2&5&8\\3&6&9\end{vmatrix}=-\begin{vmatrix}4&1&7\\5&2&8\\6&3&9\end{vmatrix} \qquad C_1 , C_2 \text{ are interchanging}$$

$$\begin{vmatrix}1&4&7\\2&5&8\\3&6&9\end{vmatrix}=-\begin{vmatrix}1&4&7\\3&6&9\\2&5&8\end{vmatrix} \qquad R_2 , R_3 \text{ are interchanging}$$

9] We can take a common factor from the elements of one row (column) outside the determinant .

$$\begin{vmatrix}10&2&9\\20&7&4\\30&8&2\end{vmatrix}=10\begin{vmatrix}1&2&9\\2&7&4\\3&8&2\end{vmatrix} \quad \& \quad \begin{vmatrix}3&4&5\\50&60&70\\8&21&3\end{vmatrix}=10\begin{vmatrix}3&4&5\\5&6&7\\8&21&3\end{vmatrix}$$

10] **Result** : To multiply the determinant by a number , we multiply one of its rows or columns by this number . $10\begin{vmatrix}3&4&5\\5&6&7\\8&21&3\end{vmatrix}=\begin{vmatrix}30&4&5\\50&6&7\\80&21&3\end{vmatrix}$

11] To add two determinants , they have to different in just one row or one column .

$$\begin{vmatrix}1&5&2\\3&7&0\\4&8&1\end{vmatrix}+\begin{vmatrix}2&6&3\\3&7&0\\4&8&1\end{vmatrix}=\begin{vmatrix}1+2&5+6&2+3\\3&7&0\\4&8&1\end{vmatrix}$$

Result :

We can write any determinant as the sum of two determinants .

$$\begin{vmatrix}1+2&5+6&2+3\\3&7&0\\4&8&1\end{vmatrix}=\begin{vmatrix}1&5&2\\3&7&0\\4&8&1\end{vmatrix}+\begin{vmatrix}2&6&3\\3&7&0\\4&8&1\end{vmatrix}$$

12] The triangle form ((Product of the elements of the main diagonal))

$$\begin{vmatrix}4&0&0\\5&1&0\\7&2&8\end{vmatrix} \quad \& \quad \begin{vmatrix}4&7&9\\0&1&6\\0&0&8\end{vmatrix}$$

Its value without expanding will be ($4\times1\times8=32$)

If the determinant is in the form $\begin{vmatrix}0&0&a_{13}\\0&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{vmatrix}$ or $\begin{vmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&0\\a_{31}&0&0\end{vmatrix}$

Its value without expanding will be ($-a_{13} \times a_{22} \times a_{31}$)

Remarks : a_{11}, a_{22}, a_{33} the elements of the main diagonals .

a_{13}, a_{22}, a_{31} **the elements of non - main diagonal .**

13] If we add any row (or any column) to another row (or column) then the value will not change .

Also

If we multiply any row (or column) by a number , and add with another row (or column) , then the value will not change .

14] Multiplying two determinants : To multiply two determinants of the same order .

$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \times \begin{vmatrix} 5 & 7 \\ 6 & 8 \end{vmatrix} = \begin{vmatrix} 1\times5+3\times6 & 1\times7+3\times8 \\ 2\times5+4\times6 & 2\times7+4\times8 \end{vmatrix} = \begin{vmatrix} 23 & 31 \\ 34 & 46 \end{vmatrix}$$

15] Cramer`s rule

$a_{11}x + a_{12}y + a_{13}z = c_1$, $a_{21}x + a_{22}y + a_{23}z = c_2$, $a_{31}x + a_{32}y + a_{33}z = c_3$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad and \quad \Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix} \quad \text{And} \quad \Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}$$

$$and \quad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix} \qquad \therefore x = \frac{\Delta_x}{\Delta} \quad , \quad y = \frac{\Delta_y}{\Delta} \quad , \quad z = \frac{\Delta_z}{\Delta}$$

Matrices

1] The matrix :

It is any array of number of elements (variables or numbers) in form horizontal rows and vertical columns written between two brackets .

2] The matrix contains m rows and n columns is of order m × n

3] $A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ ***square matrix of order*** 2×2

4] The matrix $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$ **is called square matrix of order 3 × 3**

5] $B = \begin{pmatrix} 1 & 3 & 5 & 7 \end{pmatrix}$ ***row matrix of order*** 1×4

6] $C = \begin{pmatrix}3\\4\\5\end{pmatrix}$ *column matrix of order 3 × 1*

7] $= \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$ *Zero matrix of order 2 × 2*

8] $= \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$ *Zero matrix of order 3 × 3*

9] $I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ identity matrix of order 2×2

10] $I = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$ identity matrix of order 3×3

11] *If* $A = \begin{pmatrix}2 & 3\\1 & 0\\5 & 1\end{pmatrix}$ $\therefore\ A^t = \begin{pmatrix}2 & 1 & 5\\3 & 0 & 1\end{pmatrix}$

12] $(A^t)^t = A$

13] The matrix is called symmetric if $A = A^t$

14] The matrix is called skew symmetric if $A = -A^t$

15] The skew symmetric matrix , all elements of its main diagonal must be zeroes .

Example $\begin{pmatrix}0 & 1 & -2\\-1 & 0 & 3\\2 & -3 & 0\end{pmatrix}$

16] If $A = \begin{pmatrix}1 & 4 & -10\\2 & -5 & 8\\3 & 6 & -7\end{pmatrix}$ $\therefore -5A = \begin{pmatrix}-5 & -20 & 50\\-10 & 25 & -40\\-15 & -30 & 35\end{pmatrix}$

17] If $A = \begin{pmatrix}-4 & -1\\-3 & 9\end{pmatrix}$ & $B = \begin{pmatrix}2 & -7\\8 & 2\end{pmatrix}$

$\therefore A + B = \begin{pmatrix}-4 & -1\\-3 & 9\end{pmatrix} + \begin{pmatrix}2 & -7\\8 & 2\end{pmatrix} = \begin{pmatrix}-2 & -8\\5 & 11\end{pmatrix}$

18] If $A = \begin{pmatrix}7 & -4 & 11\\6 & 5 & -1\end{pmatrix}$ & $B = \begin{pmatrix}5 & 9 & 2\\8 & -7 & -3\end{pmatrix}$

$A - B = \begin{pmatrix}7 & -4 & 11\\6 & 5 & -1\end{pmatrix} - \begin{pmatrix}5 & 9 & 2\\8 & -7 & -3\end{pmatrix} = \begin{pmatrix}2 & -13 & 9\\-2 & 12 & 2\end{pmatrix}$

19] a_{23} means the element in the **second** row and **third** column of matrix A

20] $\begin{pmatrix}3 & 5\\1 & -1\end{pmatrix}\begin{pmatrix}2 & -3 & 5\\4 & 6 & 7\end{pmatrix} =$

$\begin{pmatrix}3\times 2 + 5\times 4 & 3\times -3 + 5\times 6 & 3\times 5 + 5\times 7\\1\times 2 - 1\times 4 & 1\times -3 - 1\times 6 & 1\times 5 - 1\times 7\end{pmatrix} = \begin{pmatrix}26 & 21 & 50\\-2 & -3 & -2\end{pmatrix}$

21] $\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}3 & 5\\1 & -1\end{pmatrix}$ can't be

22] $\begin{pmatrix}1 & 2 & 3\\-1 & -2 & -3\end{pmatrix}\begin{pmatrix}2 & 3\\-2 & -3\end{pmatrix}$ can't be

23] For any matrix B $\therefore$ BI = IB = B where I is the unit matrix

24] For any three matrices A , B & C

i] A (B + C) = AB + AC ii] (AB)^t = B^{t}A^t

iii] (AB)C = A (BC) Associative

v] A (B + C) = AB + AC , (A + B) C = AC + BC Distributive

25] A + $\boxed{\quad}$ = $\boxed{\quad}$ + A (Additive identity)

26] A + (− A) = (− A) + A = $\boxed{\quad}$ (− A is the additive inverse of matrix A)

27] $A + B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 5 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$ impossible to find their sum

28] If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ *where* $\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

29] If Δ = zero $\therefore$ *The matrix is singular* .

30] Adjoint matrix

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ *then* $Adj\,(A) = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \overline{a_{31}} \\ \overline{a_{12}} & \overline{a_{22}} & \overline{a_{32}} \\ \overline{a_{13}} & \overline{a_{23}} & \overline{a_{33}} \end{pmatrix}$

31] $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 4 \\ 1 & -1 & 2 \end{pmatrix}$, *then Find Adj (A)*

The cofactor of the matrix A is

$$C = \begin{pmatrix} \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 0 & -2 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 10 & 4 & -3 \\ 2 & 4 & 1 \\ 6 & -4 & 3 \end{pmatrix}$$

$$Adj\,(A) = C^t = \begin{pmatrix} 10 & 2 & 6 \\ 4 & 4 & -4 \\ -3 & 1 & 3 \end{pmatrix}$$

32] Let A be a non singular square matrix and A^{-1}its multiplicative inverse

Then $A^{-1} = \frac{1}{|A|} = Adj\,(A)$

33] If A and B are two non singular matrices , then

a] $(AB)^{-1} = B^{-1}A^{-1}$ b] $(A^{-1})^{-1} = A$

c] $(A^{-1})^t = (A^t)^{-1}$ d] $(A^{-1})^2 = (A^2)^{-1}$

d] $(I)^{-1} = I$

Solving system of linear equations using matrix inverse

1] The matrix equation in the form $AX = B$ can be solved by using the multiplicative inverse of the square non – singular matrix i-e $|A| \neq 0$ as follow

$\because AX = B$ by multiplying both sides from the left by A^{-1}.

$\therefore A^{-1} AX = A^{-1} B$ $\quad \because A^{-1} A = I$ $\quad \therefore IX = A^{-1} B$ $\quad \therefore X = A^{-1} B$

2] The rank of the non- zero matrix is the greatest order of determinant or minor determinant of the matrix whose value does not vanish .

3] If A is a non – zero matrix of order $m \times n$, then the rank of the matrix A is denoted by RK(A) and

i] $1 \leq RK(A) \leq n$ if $m \geq n$

ii] $1 \leq RK(A) \leq m$ if $n \geq m$

$AX = B$

First : The non-homogeneous equations .

If the constant matrix B is non- zero

i-e $B \neq$ then the system represents a non - homogeneous linear equations and the system has .

1] A unique solution if $RK(A) = RK(A^*) = n$

2] Infinite number of solutions if $RK(A) = RK(A^*) = k$, where $k < n$

3] Has no solutions if $RK(A) \neq RK(A^*)$

Second : The homogeneous equations .

If the constant matrix B is a zero matrix i-e $B =$

then The system represents a homogeneous linear equations and in this case

$RK(A) = RK(A^*)$ always

i-e The rank of A is equal to the rank of augmented matrix A^* and the system has :

1] A unique solution if $RK(A) = n$, where n is the number of unknowns and the value of all variables equal zeroes , so this solution is called the zero solution (trivial solution)

3] Has infinite number of solutions other than the zero solution if $RK(A) < n$ where n is the number of unknowns

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Rule Geometry

3rd Sec .

2017

Mr.Fat7y

01002627894

Geometry

Chapter 1

1]The position vector of point A with respect to the origin is $\vec{A} = (A_x , A_y , A_z)$

A_x is called the component of $\vec{A}$ in direction of $x-$axis.

A_y is called the component of $\vec{A}$ in direction of $y-$axis.

A_z is called the component of $\vec{A}$ in direction of $z-$axis.

2]The distance between **the point** (A_x, A_y, A_z) and $x-$**axis** in the space :

Is $\sqrt{(A_y)^2 + (A_z)^2}$

3]The distance between **the point** (A_x, A_y, A_z) and $y-$**axis** in the space :

Is $\sqrt{(A_x)^2 + (A_z)^2}$

4]The distance between the point (A_x, A_y, A_z) and $z-$axis in the space :

Is $\sqrt{(A_x)^2 + (A_y)^2}$

5]The distance between the point (A_x, A_y, A_z) and xy plane in the space is $|A_z|$

6]The distance between the point (A_x, A_y, A_z) and yz plane in the space is $|A_x|$

7]The distance between the point (A_x, A_y, A_z) and xz plane in the space is $|A_y|$

8] The distance between two points in space :

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ are two points in space, then the distance between A and B is given by the relation $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

9] If $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ are two points in space .

Then the midpoint of $\overline{AB} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

10] The equation of the sphere $(x-L)^2 + (y-k)^2 + (z-n)^2 = r^2$

Where the center of the sphere (L, k, n)

and radius length $r = \sqrt{x^2 + y^2 + z^2 - d}$ where $x^2 + y^2 + z^2 > d$

$\therefore$The general equation of the sphere is : $x^2 + y^2 + z^2 - 2xL - 2yk - 2nz + d = 0$

The center of the sphere $\left(\frac{-\text{coefficient of } x}{2}, \frac{-\text{coefficient of } y}{2}, \frac{-\text{coefficient of } z}{2}\right)$

11]**The norm of a vector** :

If $\vec{A} = (A_x, A_y, A_z)$then from the distance between the point A and the origin point .

$\|\vec{A}\| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$

12]**Adding Vectors in space** :

If $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$ $\therefore \vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z)$

$\vec{A} - \vec{B} = (A_x - B_x, A_y - B_y, A_z - B_z)$

13]**Properties of adding vectors in space**:

For any two vectors $\vec{A}$ and $\vec{B} \in R^3$, then:

a]**Closure property** : $\vec{A} + \vec{B} \in R^3$

b]**Commutative property** : $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

c]**Associative property** : $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

d] Identity element over addition (zero vector) : $\vec{0} = (0, 0, 0)$ is the neutral element of addition in R^3

$\therefore \vec{A} + \vec{0} = \vec{0} + \vec{A} = \vec{A}$

e]**The additive inverse** : For every vector $\vec{A} = (A_x, A_y, A_z) \in R^3$ there is

$-\vec{A} = (-A_x, -A_y, -A_z) \in R^3$ such that : $\vec{A} + (-\vec{A}) = (-\vec{A}) + \vec{A} = \vec{0}$

14] **Multiplying a vector by a scalar (a real number)**

For any vector $\vec{A} \in R^3$ and K $\in$ R $\therefore K\vec{A} = K(A_x, A_y, A_z) = (kA_x, kA_y, kA_z) \in R^3$

i] $3(2, -1, 4) = (6, -3, 12)$ ii] $-2(1, -3, -4) = (-2, 6, 8)$

15] For any two vector $\vec{A}, \vec{B} \in R^3$ and $K \in R$ $\therefore K(\vec{A} + \vec{B}) = K\vec{A} + K\vec{B} \in R^3$

For any vector $\vec{A} \in R^3$ and k, $L \in R$ $\therefore (k + L)\vec{A} = k\vec{A} + L\vec{A}$ & $k(L\vec{A}) = L(k\vec{A}) = (kL)\vec{A}$

16] **Equality of vectors in the space:**

If $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z)$ $\therefore \vec{A} = \vec{B}$ **if and only if** : $A_x = B_x$, $A_y = B_y$, $A_z = B_z$

17]**The unit vector in a direction of a given vector** :

If $\vec{A} = (A_x, A_y, A_z) \in R^3$, then the unit vector in the direction of the vector $\vec{A}$ is denoted by $\overrightarrow{U_A}$ and given by the relation : $\overrightarrow{U_A} = \frac{\vec{A}}{\|\vec{A}\|}$ (is a vector whose norm equals the unit length)

18] **Fundamental unit vectors** : $(\hat{i}, \hat{j}, \hat{k})$

$\hat{i} = (1, 0, 0)$ The unit vector in the +ve direction of x-axis

$\hat{j} = (0, 1, 0)$ The unit vector in the +ve direction of y-axis

$\hat{k} = (0, 0, 1)$ The unit vector in the +ve direction of z-axis

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19] **Expressing a vector in terms of the fundamental unit vectors :**

$\vec{A} = (A_x , A_y , A_z) = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

20] If $\vec{A} = (A_x , A_y , A_z)$ is a vector in space and $(\theta_x , \theta_y , \theta_z)$ are the measures of the angles made by the vector with the + ve directions of x , y , z axes respectively , then :

$A_x = \|\vec{A}\| Cos\,\theta_x$, $A_y = \|\vec{A}\| Cos\,\theta_y$, $A_z = \|\vec{A}\| Cos\,\theta_z$

$(\theta_x , \theta_y , \theta_z)$ **are called the direction angles of vector** $\vec{A}$

$Cos\,\theta_x , Cos\,\theta_y , Cos\,\theta_z$ **are called the direction cosines of vector** $\vec{A}$.

21] **Notice that :** $Cos\,\theta_x\hat{i} + Cos\,\theta_y\hat{j} + Cos\,\theta_z\hat{k}$

represent the unit vector in the direction of the vector $\vec{A}$. i.e. $Cos^2\theta_x + Cos^2\theta_y + Cos^2\theta_z = 1$

22] If $\vec{A}$ and $\vec{B}$ are two vectors in R^3. Is $\|\vec{A} + \vec{B}\| = \|\vec{A}\| + \|\vec{B}\|$?

Which side is greater if both sides are unequal ?

The Solution

For any two vectors $\vec{A} = (A_x , A_y , A_z)$, $\vec{B} = (B_x , B_y , B_z) \in R^3$

$\therefore \|\vec{A}\| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$, $\|\vec{B}\| = \sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2}$

$\|\vec{A}\| + \|\vec{B}\| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2} + \sqrt{(B_x)^2 + (B_y)^2 + (B_z)^2}$

$\|\vec{A} + \vec{B}\| = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2 + (A_z + B_z)^2}$ $\therefore \|\vec{A}\| + \|\vec{B}\| \le \|\vec{A} + \vec{B}\|$

23] The scalar product of two vectors (Dot product)

$\vec{A}.\vec{B} = \|\vec{A}\|\,\|\vec{B}\| Cos\,\theta$ $\therefore$ $Cos\,\theta = \dfrac{\vec{A}\bullet\vec{B}}{\|\vec{A}\|\,\|\vec{B}\|}$

θ is the measure of angle between two non zero vectors where $0^0 \le \theta \le 180^0$

Special cases:

i] **If** $Cos\,\theta = 1$, **then** $\vec{A} , \vec{B}$ are parallel and in the same direction.

ii] **If** $Cos\,\theta = -1$, **then** $\vec{A} , \vec{B}$ are parallel and in the opposite direction.

iii] **If** $Cos\,\theta = 0$, **then** $\vec{A} , \vec{B}$ are perpendicular.

24] $(\hat{\imath}, \hat{\jmath}, \hat{k})$ *are right system of unit vectors*

a] $\hat{\imath} \bullet \hat{\imath} = \|\hat{\imath}\| \times \|\hat{\imath}\| \times \cos 0^0 = 1$

b] $\hat{\jmath} \bullet \hat{\jmath} = \|\hat{\jmath}\| \times \|\hat{\jmath}\| \times \cos 0^0 = 1$

c] $\hat{k} \bullet \hat{k} = \|\hat{k}\| \times \|\hat{k}\| \times \cos 0^0 = 1$

d] $\hat{\imath} \bullet \hat{\jmath} = \|\hat{\imath}\| \times \|\hat{\jmath}\| \times \cos 90^0 = 0$

e] $\hat{\jmath} \bullet \hat{k} = \|\hat{\jmath}\| \times \|\hat{k}\| \times \cos 90^0 = 0$

f] $\hat{\imath} \bullet \hat{k} = \|\hat{\imath}\| \times \|\hat{k}\| \times \cos 90^0 = 0$

****

25] **Properties of the scalar product**:

a] $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$ Commutative property

b] $\vec{A} \bullet \vec{A} = \|\vec{A}\|^2$

c] $\vec{A} \bullet \vec{B} = 0$ **If and only if $\vec{A}, \vec{B}$ are perpendicular (condition of perpendicularity)**

d] $\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$ distributive property

e] $(k\vec{A}) \bullet \vec{B} = \vec{A} \bullet (k\vec{B}) = k(\vec{A} \bullet \vec{B})$ **K is a real number**

****

26]The scalar product of two vectors in an orthogonal coordinate system :

For any two vectors $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (B_x, B_y, B_z) \in R^3$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

Remember that :

For any two vectors $\vec{A} = (A_x, A_y)$, $\vec{B} = (B_x, B_y) \in R^2$ $\therefore \vec{A} \bullet \vec{B} = A_x B_x + A_y B_y$

****

27] **If** $\vec{A}$ and $\vec{B}$ are vectors , **then** the component of vector $\vec{A}$ is in the direction $\vec{B}$

(denoted A_B) is $A_B = \|\vec{A}\| \cos\theta = \dfrac{\vec{A} \bullet \vec{B}}{\|\vec{B}\|}$

****

28] If a force $\vec{F}$acts on a body to move it a displacement $\vec{S}$, we say that the force does work which can be found by the relation: $W = \vec{F} \bullet \vec{S} = \|\vec{F}\| \, \|\vec{S}\| \cos\theta$

The unit of measuring the work is the unit of measuring the force $\times$ unit of measuring the displacement.

Special cases :

a] If the force $\vec{F}$ is in the direction of the displacement $\vec{S}$ $(\theta = 0^0)$

$\therefore W = \|\vec{F}\| \, \|\vec{S}\|$

b] If the force $\vec{F}$ is in the opposite direction of the displacement $(\theta = 180^0)$

$\therefore W = -\|\vec{F}\|\,\|\vec{S}\|$

c] If the force $\vec{F}$ is perpendicular to the direction of the displacement ($\theta = 90^0$)

$\therefore W == 0$

****

29] **If** $\vec{A}$ and $\vec{B}$ two vectors in a plane the measure of the smallest angle between them is θ and $\vec{C}$ is a unit vector perpendicular to the plane which contains $\vec{A}$and $\vec{B}$, **then** the cross product of $\vec{A}$ and $\vec{B}$ is given by the relation . $\vec{A} \times \vec{B} = (\|\vec{A}\|\,\|\vec{B}\| Sin\,\theta)\,\vec{C}$

The direction of the unit vector $\vec{C}$ is defined (up or down) According to the right hand rule where the curved fingers of the right hand show the direction of the relation from the vector $\vec{A}$ to the vector $\vec{B}$, then the thump shows the direction of the vector $\vec{C}$.

****

30] **Important notes** :

a] $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$

b] $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$

c] $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$

d] $\vec{A} \times \vec{A} = \vec{0}$

e] If $\vec{A} \times \vec{B} = \vec{0}$then either $\vec{A} /\!/ \vec{B}$ or one of them or both is equal to $\vec{0}$

in this case : $\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}$ $\therefore \vec{A} = k\,\vec{B}$

When k > 0 , the two vectors are parallel and in the same direction and when k < 0 , the two vectors are parallel and in opposite directions.

f] $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$ distribution property

g] $(m\vec{A}) \times \vec{B} = (\vec{A} \times m\vec{B}) = m(\vec{A} \times \vec{B})$where m is a real number

****

31] If $\vec{A} = (A_x , A_y , A_z)$, $\vec{B} = (B_x , B_y , B_z)$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{\imath} - (A_zB_x - A_xB_z)\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$$

****

32] Special case : If $\vec{A} = (A_x , A_y)$, $\vec{B} = (B_x , B_y)$

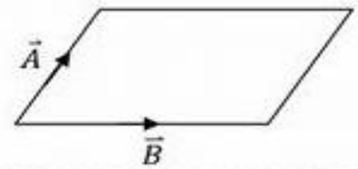
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & 0 \\ B_x & B_y & 0 \end{vmatrix} = (A_xB_y - A_yB_x)\hat{k}$$

****

33]The geometric mean of vector product $\vec{A} \times \vec{B} = (AB \sin\theta)\vec{C}$,

$\|\vec{A} \times \vec{B}\|$ =area of parallelogram whose two sides $\vec{A}$ and $\vec{B}$.

= 2 area of triangle whose two sides $\vec{A}$ and $\vec{B}$



34] **The scalar triple product** :

If $\vec{A}, \vec{B}, \vec{C}$ are vectors , then the expression $\vec{A} \cdot \vec{B} \times \vec{C}$ is known as the scalar triple product which has a lot of applications in the statics filed (notice that the expression has no brackets where doing the scalar product first is meaningless)

$$\vec{A} \cdot \vec{B} \times \vec{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

35] The value of scalar triple product does not change if the vectors are permuted in such a way that they are still read as the same cyclic order.

$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B}$ **notice the cyclic order of $\vec{A}, \vec{B}, \vec{C}$**

36] If the vectors $\vec{A}, \vec{B}, \vec{C}$ in the same plane , then the scalar triple product vanishes

So $\vec{A} \cdot \vec{B} \times \vec{C} = 0$

37]**The geometrical meaning of the scalar triple product** :

If $\vec{A}, \vec{B}, \vec{C}$ are 3 vectors forming 3− non parallelsides of a parallelepiped , then the volume of the parallelepiped = the absolute value of the scalar triple product.

∴The volume of the parallelepiped = $|\vec{A} \cdot \vec{B} \times \vec{C}|$

We use $|\vec{A} \cdot \vec{B} \times \vec{C}| = 0$ to prove that $\vec{A}, \vec{B}, \vec{C}$ are in the same plane .

Chapter 2

Straight Lines and planes in space

1] $\theta_x , \theta_y , \theta_z$ are the directed angles of a straight line in space

$\therefore$ Cos θ_x , Cos θ_y , Cos θ_z are directed cosines of these straight lines and they are usually denoted by ℓ , m , n . where ℓ = Cos θ_x , m = Cos θ_y , n = Cos θ_z $\therefore \ell^2 + m^2 + n^2 = 1$

$\therefore \vec{U} = \ell\,\hat{\imath} + m\,\hat{\jmath} + n\,\hat{k}$ is the unit vector in the direction of the straight line

And any vector parallel to the unit vector $\vec{U}$ is called the direction vector of the straight line and is denoted by $\vec{d}$ where $\vec{d} = t(\ell , m , n) = (a , b , c)$

a , b , c are called direction ratio (direction numbers)

The direction vector of the straight line takes different equivalent forms such as

$\vec{d} = 2(\ell, m, n) = 3(\ell , m, n) = -4(\ell, m, n)$ ……………………

2] **Vector form of the equation of a straight Line in space** : $\vec{r} = \vec{A} + t\,\vec{d}$

If L is a straight line in space whose direction vector is $\vec{d} = (a , b , c)$ and passes through the point A whose position vector is $\vec{A} = (x_1 , y_1 , z_1)$

$\therefore (x , y , z) = (x_1 , y_1 , z_1) + t(a , b , c)$

3] Parametric equations of a straight Line in space :

$$(x = x_1 + at \ , \ y = y_1 + bt \ , \ z = z_1 + ct)$$

4] The Cartesian equation of a straight Line in space $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

Special cases :

a] In the case a = 0 (say) , then the Cartesian form of the equation of the straight line takes the form of $x = x_1$, $\frac{y - y_1}{b} = \frac{z - z_1}{c}$

b] Since the direction ratios a , b , c are proportion to the direction cosine ℓ , m , n , then we can write the Cartesian form of the equation of the straight line in the form

$$\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

5] The angle between two straight lines in space

If L_1 , L_2 are two straight lines in space whose direction vectors are $\overrightarrow{d_1} = (a_1, b_1, c_1)$ and $\overrightarrow{d_2} = (a_2, b_2, c_2)$, then the smallest angle between the two straight lines L_1 , L_2 is given by the relation :

$$\cos\theta = \frac{|\overrightarrow{d_1} \bullet \overrightarrow{d_2}|}{\|\overrightarrow{d_1}\|\ \|\overrightarrow{d_2}\|} \ , \ 0 \le \theta \le \frac{\pi}{2}$$

and if (ℓ_1, m_1, n_1) , (ℓ_2, m_2, n_2) are the direction cosines for the two Straight lines , then: $\cos\theta = |\ell_1\ell_2 + m_1m_2 + n_1n_2|$

****

6] * If the two straight lines are parallel and there is a point on one of them satisfying the equation of the other , then the two straight lines are coincident

** Any two straight lines are (parallel **or** intersect **or** skew)

****

The equation of a plane in space

7] If point A (x_1, y_1, z_1) belongs to the plane and its position vector is $\vec{A}$ and the normal direction vector to the plane is $\vec{n} = (a, b, c)$ and B (x, y, z) any point on the plane its position vector is $\vec{r}$ $\therefore \vec{n} \bullet \overrightarrow{AB} = 0$ $(\vec{r} = \vec{B})$

$\therefore \vec{n} \bullet \vec{r} = \vec{n} \bullet \vec{A}$ the vector form of the equation of the plane .

Remark : To find the vector equation of the plane , we must know a point on the plane and the perpendicular direction vector to the plane.

****

8] The standard form and general form of the equation of a plane in space

From the vector form of the equation of the plane $\vec{n} \bullet (\vec{r} - \vec{A})$

$\vec{n} = (a, b, c)$, $\vec{r} = (x, y, z)$, $\vec{A} = (x_1, y_1, z_1)$

$(a, b, c) \bullet (x - x_1, y - y_1, z - z_1)$ the standard form for the equation of the plane

$ax + by + cz + (-a x_1 - b y_1 - c z_1) = 0$ Let $-a x_1 - b y_1 - c z_1 = d$

$\therefore ax + by + cz + d = 0$ The general form of the equation of the plane

****

9] The angle between two planes .

The measure of the angle between two planes is the measure of the angle between their two normal vectors i.e. $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ are the two normal vectors on the two planes , then the measure of the angle between the two planes is given by the relation

$$\text{Cos } \theta = \frac{|\overrightarrow{n_1} \cdot \overrightarrow{n_2}|}{\|\overrightarrow{n_1}\| \ \|\overrightarrow{n_2}\|} \quad \text{where} \quad 0 \le \theta \le 90^0$$

****

10] Parallel planes and perpendicular planes .

If $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ are the normal vectors to the two planes , then

i] The two planes are **parallel** if $\overrightarrow{n_1} \parallel \overrightarrow{n_2}$ i-e $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

ii] The two planes are **perpendicular** $\overrightarrow{n_1} \cdot \overrightarrow{n_2}$ i-e $a_1a_2 + b_1b_2 + c_1c_2 = 0$

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11] The length of the perpendicular drawn from a point to a plane .

If $A(x_1, y_1, z_1)$ is a point outside the plane x and B is a point on the plane , $\vec{n}$ is the normal vector to the plane , then the distance from the point A to the plane equals the length of the projection of $\overrightarrow{BA}$ to $\vec{n}$. $\therefore$ $L = \frac{|\overrightarrow{BA} \cdot \vec{n}|}{\|\vec{n}\|}$

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12] Cartesian form of the perpendicular length drawn from a point and a plane

You notice that the perpendicular length from point $A(x_1, y_1, z_1)$ to the plane passing through $B(x_2, y_2, z_2)$ and $\vec{n} = (a, b, c)$ is the normal vector to the plane is given by the relation $L = \frac{|\overrightarrow{BA} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|(x_1-x_2, y_1-y_2, z_1-z_2) \cdot (a, b, c)|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_1+by_1+cz_1+(-ax_2-by_2-cz_2)|}{\sqrt{a^2+b^2+c^2}}$

Point $B(x_2, y_2, z_2)$ lies on the plane $a x + b y + c z + d = 0$

$\therefore -ax_2 - by_2 - cz_2 = d$ $\quad \therefore L = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

The Cartesian form of the length of the perpendicular

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13] Equation of a plane using the intercepted parts from the coordinate axes

If a plane cuts the coordinates axes at points $(x_1, 0, 0), (0, y_1, 0) (0, 0, z_1)$, then the equation of the plane is in the form $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$

The equation of the plane in terms of the intercepted parts from the coordinate axes

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14] The general equation of the plane passes through (x_1, y_1, z_1) ' (x_2, y_2, z_2) ' (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$

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15] The surface area of the triangle $\Delta = \sqrt{P(P-a)(P-b)(P-c)}$

Heron's formula . **Where** $P = \frac{1}{2}(a + b + c)$

Example] Find the surface area of the triangle whose side lengths are **6** , **8** and **10** centimeters using **Heron's formula** .

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